VARIETIES OF SOLUBLE GROUPS

by

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The work presented in this thesis is my own except where otherwise stated.

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CONTENTS

CHAPTER 1

INTRODUCTION

In 1954, in the first of a well known series of papers on infinite soluble groups, P. Hall [8] demonstrated the existence of a dichotomy which distinguishes the property 'max-n' for finitely generated soluble groups. In subsequent work, he went on to introduce and discuss other dichotomies distinguishing group-theoretical properties. The aim of this thesis will be to extend some of these dichotomies, using the methods and results of the theory of varieties of groups.

Hall described his dichotomies in terms of what he called commutator-subgroup functions. Since, however, our interests will lie with varieties, and since the language of varieties is probably more natural in this context, we shall rephrase the result in varietal terms. Hall defined then, in [8], a special class of soluble varieties of groups (which we shall call the class of Hall varieties), which is precisely the class of varieties obtainable from the trivial variety by commutation. He then noted that a Hall variety is either abelian by nilpotent or contains the variety of all centre by metabelian groups. Finally, he showed that all finitely generated abelian by nilpotent groups have max-n and that there are finitely generated centre by metabelian groups that do not have max-n . Hence, in this way, this dichotomy of Hall varieties distinguishes the property 'max-n' .

Consider the partially ordered set of Hall varieties. If P is a property of Hall varieties which is inherited by subvarieties, then those varieties with P will form an ideal and those which do not satisfy P will form a dual ideal. It is not difficult to see, in fact, that this partially ordered set has the minimal condition, and so that each dual ideal is generated by its minimal elements. In this language, the results of Hall show that, if P is the property 'locally max-n' , then the Hall variety of all centre by metabelian groups is the only minimal non-P element and that the ~~dual~~ ideal of Hall varieties with P is generated by the abelian by nilpotent Hall varieties.

When we attempt to establish results of this kind for arbitrary varieties of soluble groups, as we shall do in this thesis, we do not retain the minimal condition, and so we can no longer guarantee the existence of any, let alone enough, minimal elements of the type we describe. However, the special conditions of most of the problems we consider allow us to show that the relevant dual ideals are generated by their minimal elements. (In fact, we show in 2.3.10 that it is sufficient that the relevant ideal be generated by a set of finitely based varieties.) We remark that the essential problem we encounter in extending these dichotomies of Hall, is to establish suitable generators for these ideals and dual ideals. It then usually follows from known group-theoretical results that the dichotomy distinguishes the relevant property.

Before we describe our results, we shall require some notation. We denote by $\underline{\underline{A}}_n$ the varieties of all abelian groups of exponent n (where if $n = 0$, we write $\underline{\underline{A}}_0$ as $\underline{\underline{A}}$). We denote by $\underline{\underline{N}}_c$ the variety of all nilpotent groups of class at most c and by $\underline{\underline{C}}_p$ the VARIETY GENERATED BY THE RELATIVELY FREE GROUP OF RANK 2 OF THE variety of all centre by metabelian groups with derived group of prime exponent p (or exponent 4, if $p = 2$). The product of two varieties will be denoted, as usual, by juxtaposition. The class of finitely generated groups will be written G and the class of torsion-free groups T.

The first dichotomy of Hall varieties which we shall consider is that which divides Hall varieties into the classes $\{\underline{\underline{N}}_c \mid c$ is a natural number$\}$ and $\{\underline{\underline{V}} \mid \underline{\underline{V}} \geq \underline{\underline{A}}^2\}$ $(\underline{\underline{A}}^2 = \underline{\underline{A}}\underline{\underline{A}})$. Hall noted this dichotomy in [11] and pointed out that it distinguishes, for finitely generated soluble groups, each of the properties of nilpotence, finite presentation, polycyclicity or max (the last two, of course, being equivalent for soluble groups).

The initial impetus for the study of extensions of this dichotomy came from the dichotomies that emerged from the known results on metabelian varieties (in particular, the classification of metabelian varieties of exponent zero, due to L.G. Kovács and M.F. Newman, given here as 2.3.3 and 2.3.4). It appeared that the varieties $\underline{\underline{A}}_p\underline{\underline{A}}_q$, $\underline{\underline{A}}_p\underline{\underline{A}}$, $\underline{\underline{A}}\underline{\underline{A}}_p$, $\underline{\underline{A}}^2$ (p and q not necessarily distinct primes) determine particularly important dichotomies and the question naturally arose as to whether they also determine similar dichotomies for soluble varieties in general. In this direction, A.L. Šmel'kin [26] had shown that $\underline{\underline{A}}^2$ retained its role, at least for nilpotent by abelian varieties, for such a variety is finite exponent by nilpotent by finite exponent precisely if it does not contain $\underline{\underline{A}}^2$. Also, it is well known that any soluble variety is nilpotent precisely if it does not contain any of the varieties $\underline{\underline{A}}_p\underline{\underline{A}}_q$. (It is interesting, although possibly irrelevant, to note that the varieties listed above are precisely the metabelian varieties which are products of two irreducible varieties.)

We shall describe our extensions of this Hall dichotomy in the following form:

we list, for each extension, three classes $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. $\mathcal{P}$ is a property of a soluble variety; $\mathcal{Q}$ is a set of generators for the ideal of varieties with $\mathcal{P}$; $\mathcal{R}$ is a set of minimal non-$\mathcal{P}$ varieties which generates the relevant dual ideal.

Thus each result may be stated as:

let $\underline{\underline{V}}$ be a soluble variety. Then $\underline{\underline{V}}$ is a subvariety of a variety of $\mathcal{Q}$ and has $\mathcal{P}$, if and only if $\underline{\underline{V}}$ contains no variety from $\mathcal{R}$.

Then we have the following, where k, ℓ and c are natural numbers, p and q are prime numbers, and when we write, for example, $\underline{\underline{A}}_k^{\ell}\underline{\underline{N}}_c\underline{\underline{A}}_k^{\ell}$, we take this to mean the set of all varieties obtainable by allowing k, ℓ and c to take all possible values.

	$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$
A)	$G \wedge T \wedge \underline{\underline{V}}$ has max	$\underline{\underline{A}}_k^{\ell}\underline{\underline{N}}_c\underline{\underline{A}}_k^{\ell}$	$\underline{\underline{A}}^2$, ?
A')	$G \wedge \underline{\underline{V}}$ has max	$\underline{\underline{N}}_c\underline{\underline{A}}_k^{\ell}$	$\underline{\underline{A}}_p\underline{\underline{A}}$
B)	$T \wedge \underline{\underline{V}}$ is nilpotent	$\underline{\underline{A}}_k^{\ell}\underline{\underline{N}}_c$	$\underline{\underline{A}}\underline{\underline{A}}_p$
B')	$\underline{\underline{V}}$ is nilpotent	$\underline{\underline{N}}_c$	$\underline{\underline{A}}_p\underline{\underline{A}}_q$

(The last result is well known - we do not prove it in this thesis and include it here only because it appears to belong with the results we have proved.)

In the case of A), we have not been able to establish that $\underline{\underline{A}}^2$ is the only variety in R. In Chapter 4, however, we shall show that any other variety in R has the following properties:

i) it is not generated by the finite groups it contains;

ii) it is a subvariety of $\underline{\underline{AN_2A}}$ *;*

iii) it is not metanilpotent.

The 'pathological' nature of such a variety depends, of course, largely on *i)*; it appears to be an open question whether soluble varieties satisfying *i)* exist at all. We note that Narain Gupta [6] has also proved *iii)*; his methods being quite different. We shall prove A') and B) in Chapter 3. In fact we shall show that similar dichotomies hold in an even more general situation than that of soluble varieties alone. Finally, we note that the property 'max' may be replaced by 'finitely related' in the statements of A) and A').

The second dichotomy of Hall varieties we consider is the one which distinguishes max-n . Although this one involves considerably more complication than the previous one and, consequently, our results are less complete, there seems to be no reason why a set of results paralleling those we have given above should not hold. We shall, however, consider only the problem parallel to A'); that is, we shall be interested in describing a dichotomy which distinguishes the property 'locally max-n' (we denote this property by $\mathcal{P}$). As before, to describe this dichotomy we must identify the minimal non-$\mathcal{P}$ varieties, show that they generate the dual ideal of non-$\mathcal{P}$ varieties, and also find a suitable generating set for the ideal of varieties with $\mathcal{P}$.

The first problem, then, is to establish a likely set of minimal non-$\mathcal{P}$ varieties. For this, we turn to the examples of finitely generated soluble groups which do not have max-n , given by Hall in [8]. He gives, for each prime p , an example of such a group in $\mathbf{C}_p$. Also it appears that, among the examples of finitely generated soluble groups which do not have max-n that are known to us, each generates a variety that contains one of these examples of Hall. Thus the $\mathbf{C}_p$ appear to be a reasonable first guess as the set of minimal non-$\mathcal{P}$ varieties.

In Chapter 5, we shall show that this choice is correct at least for soluble, centre extended by abelian by nilpotent by finite exponent varieties. That is, we shall show that, the dual ideal of non-$\mathcal{P}$ varieties in the partially ordered set of such varieties is generated by its minimal elements, which are just the $\mathbf{C}_p$, and that the set $\left\{\mathbf{A}\mathbf{N}_c\mathbf{A}_k^{\ell} \mid c, k, \ell \text{ are natural numbers}\right\}$ generates the ideal of varieties with $\mathcal{P}$. Evidently, having established this dichotomy, we may replace the property 'max-n' by any other group-theoretical property (for example, 'residual finiteness'), which is known to hold for finitely generated abelian by nilpotent by finite groups and for which there are counterexamples in each $\mathbf{C}_p$.

These results extend some theorems of C.K. Gupta, N.D. Gupta and A.H. Rhemtulla [5]. In this paper, they show, for centre by metabelian varieties, the first half of our result above; that is, they show that the $\underline{\underline{C}}_p$ are generators of the dual ideal dichotomising either max-n or residual finiteness. (Our results were obtained largely independently of theirs, although we are indebted to them for a technique we use in Chapter 5.) In Chapter 6, we extend their results in a different direction by showing that they are, in fact, true for all metanilpotent varieties. To rescue these results from the rather complicated language in which they are, by now, framed, we show that all the finitely generated groups of a metanilpotent variety will have max-n precisely if the variety contains none of the varieties $\underline{\underline{C}}_p$.

Finally, we point out that there are many more easily established dichotomies of Hall varieties. It would be surprising if these were not to prove useful in the study of soluble varieties in general.

(The results of Chapter 4 have appeared in my paper, "On varieties of soluble groups", *Bull. Austral. Math. Soc.* 5 (1971), 95-109.)

CHAPTER 2

NOTATION AND PRELIMINARY RESULTS

2.1 Notation

The basis for the notation and terminology of this thesis will be Hanna Neumann's book [19]. In particular, any unexplained term will have the meaning ascribed to it there. However, owing to alterations in current usage, and, in some cases, to personal preference, there will be a number of differences between our notation and that of [19]. Therefore, in this section and in the preliminary work of the following two sections, we shall briefly outline the main details of notation and terminology that we shall use.

We shall use capital Roman letters for groups and small Roman letters for elements of groups. E will represent every trivial group and e the identity element of every group. (Occasionally, such as when working with the additive group of a module, we shall have occasion to use additive notation and then E, e will be replaced by 0, o .) Doubly underlined Roman capitals (to replace the German capitals used in [19]) will be used to represent varieties and the trivial variety will be denoted $\underline{\underline{E}}$.

We shall use $\underline{Q}$ for the field of rational numbers, $\underline{Z}$ for the ring of rational integers and $\underline{N}$ for the set of natural numbers (by which we mean the set of positive integers). If $n \in \underline{N}$, $\underline{Z}/(n)$ will denote the ring of residues modulo n . If n is a prime number, p say, then $\underline{Z}/(p)$ is the Galois field with p elements which we shall denote $\underline{GF}(p)$. The multiplicative identity of all these entities will be represented by 1 . Finally, mappings and functions will be represented by small Greek letters.

2.2 Groups

If G is a group and H is a subgroup of G , we write $H \leq G$; if H is a normal subgroup of G , we write $H \trianglelefteq G$. The index of H in G is denoted by $|G : H|$. If Δ is a set with elements either subgroups, subsets or elements of G , then we shall write $gp_G(\Delta)$ $\{$respectively, $nsgp_G(\Delta)\}$ for the subgroup of G (respectively, the normal subgroup of G) generated by the elements of Δ . Where no confusion is likely to arise, we shall omit the subscript G . We shall say that G is torsion-free if no element of G , except the identity, has finite order.

If g, h are elements of G , we shall write the conjugate $h^{-1}gh$ of g by h as g^h and the commutator $g^{-1}h^{-1}gh$ as $[g, h]$. If H, K are subgroups of G , then $[H, K]$ will denote $gp_G(\{[h, k] \mid h \in H, k \in K\})$. If $g_1, \ldots, g_n$ are elements of G , we define the left-normed commutator $[g_1, \ldots, g_n]$ of weight n inductively by

$$[g_1] = g_1$$

$$[g_1, \ldots, g_n] = \left[[g_1, \ldots, g_{n-1}], g_n\right] ;$$

and similarly we define $[H_1, \ldots, H_n]$, where $H_1, \ldots, H_n$ are subgroups of G , by

$$[H_1] = H_1$$

$$[H_1, \ldots, H_n] = \big[[H_1, \ldots, H_{n-1}], H_n\big] .$$

The expression $[g_1, cg_2]$ $(c \in \underline{N})$ is defined inductively by $[g_1, 1g_2] = [g_1, g_2]$ and $[g_1, cg_2] = \big[[g_1, (c-1)g_2], g_2\big]$ if $c > 1$; again a similar notation holds for $[H_1, cH_2]$. As we shall use commutator identities largely under special conditions, we shall introduce most of them later, as we need them. We do, however, note three of the most widely used ones here:

if $g, h, k \in G$, then

$$[gh, k] = [g, k][g, k, h][h, k] = [g, k]^h[h, k]$$

$$[g, hk] = [g, k][g, h][g, h, k] = [g, k][g, h]^k$$

$$[g, h] = [h, g]^{-1} .$$

If $H \leq G$, the centraliser of H in G will be written $C_G(H)$ and we shall abbreviate $C_G(G)$ - that is, the centre of G - by $Z(G)$. The terms of the upper central series of G will be written $Z_i(G)$ $(i \in \underline{N})$ - so that $Z_1(G) = Z(G)$. It is easy to see that an element g of G lies in $Z_i(G)$ if and only if $[g, g_1, \ldots, g_i] = e$ for every $g_1, \ldots, g_i$ in G . We shall denote the $(i+1)$th term of the lower central series by $\underline{\underline{N}}_i(G)$ and the i'th term of the derived series by $\underline{\underline{A}}^i(G)$ $\Big($so that $G = \underline{\underline{N}}_0(G) = \underline{\underline{A}}^0(G)\Big)$. We shall usually denote the derived group of

G, however, by G' rather than use either of the two notations suggested above.

Suppose that A is an abelian normal subgroup of G. Then G acts on A by conjugation and, in the usual way, this gives A a structure as a $\underline{Z}G$-module $\left(\text{in fact, even as a } \underline{Z}\left(G/C_G(A)\right)\text{-module}\right)$. Thus, if $a \in A$, $k_1, \ldots, k_n \in \underline{Z}$, and $g_1, \ldots, g_n \in G$, then $k_1g_1 + \ldots + k_ng_n \in \underline{Z}G$ and $a^{k_1g_1+\ldots+k_ng_n}$ will represent $\prod_{i=1}^{n} a^{k_ig_i}$. If, further, A has finite exponent n, A has a $\left(\underline{Z}/(n)\right)G$-module structure and, of course, if A has prime exponent p, A has a $\underline{GF}(p)G$-module structure. We shall frequently use the notation $a^{k_1g_1+\ldots+k_ng_n}$ without further reference to the module structure, apart from an indication as to which of the above sets the k_i belong to when there is a possibility of ambiguity.

If G is an arbitrary group, we define the Fitting subgroup of G to be the product of all the nilpotent normal subgroups of G. Although the following lemma is well known, we include a proof for purposes of comparison with the similar, but lesser known, result of 2.3.2.

2.2.1 LEMMA. *If* G *is a soluble group and* N *is its Fitting subgroup, then* $C_G(N) \leq N$.

Proof. Write $C = C_G(N)$ and suppose that $C \nleq N$. Then CN/N is non-trivial and soluble. Thus there is a normal subgroup A of

G such that $N < A \leq CN$ and A/N is abelian. Hence $A = A \cap CN = (A \cap C)N$ (using the modular law for normal subgroups). Then, if $B = A \cap C$, $B \trianglelefteq G$ and $[B', B] \leq [N, C] = E$. Thus B is nilpotent and so $B \leq N$. Hence $A = (A \cap C)N = BN = N$, contradicting the choice of A and so proving the lemma.

The following two lemmas give criteria for the nilpotency of a group. The first is a sharp, in fact the best possible, version of the well known result of P. Hall [10]; it is due to A.G.R. Stewart [27].

2.2.2 LEMMA. *Suppose that a normal subgroup* N *of a group* G *is nilpotent of class* c *and that* G/N' *is nilpotent of class* d. *Then* G *is nilpotent of class at most* $cd + (c-1)(d-1)$.

The second of these two lemmas, which may be regarded as the torsion-free counterpart of 2.2.2, appears as Lemma 6 in A.L. Šmel'kin [26]: it is described there as well known and included without proof. Since we are not aware of a proof in the literature, we include one here.

If N is a nilpotent, torsion-free group and L is a subgroup of N, we shall say that L is isolated if whenever $x \in N$ and, for some $n \in \underline{N}$, $x^n \in L$, then $x \in L$. It is then not difficult to show that the intersection of isolated subgroups is isolated and so that, for each subgroup M of N, there is a least isolated subgroup of N containing M, which we shall call the isolator of M in N and write $I_N(M)$. If M is normal in N, then $I_N(M)/M$ is evidently the torsion subgroup of N/M and so $I_N(M)$ is also normal in M. Similarly if M is characteristic (respectively, fully invariant) in N, then so also is $I_N(M)$.

2.2.3 LEMMA. *Suppose that* N *is a torsion-free nilpotent group of class* c, *that* N *is a normal subgroup of a group* G *and that* $G/I_N(N')$ *is nilpotent of class* d. *Then* G *is nilpotent of class at most* $cd + (c-1)(d-1)$.

Proof. We form the Mal'cev completion N^* of N (see Theorem 4.8 of Michel Lazard [15] for a definition of N^* and a proof of its existence). Then N^* is also nilpotent of class c (see, for example, Corollary 15.3 of Gilbert Baumslag [1]). We shall also require the following result, which follows immediately from Theorem 4.10 of Michel Lazard [15]:

There exists a monomorphism $\alpha : \mathrm{Aut}\, N \to \mathrm{Aut}\, N^*$ such that, if $u \in \mathrm{Aut}\, N$, then the restriction of $u\alpha$ to N coincides with u.

Now G, as an extension of N by G/N, is determined by a map $\rho : G/N \to \mathrm{Aut}\, N$ and a factor set $\{(u, v) \mid u, v \in (G/N)\cancel{\rho}\}$ (see, for example, 15.1.1 of Marshall Hall [7]). Then we consider the map $\rho\alpha : G/N \to \mathrm{Aut}\, N^*$ and the factor set $\{(u, v)\eta \mid u, v \in (G/N)\cancel{\rho\alpha}\}$ WHERE η IS THE EMBEDDING $\eta : N \to N^*$. ~~defined by $(u\alpha, v\alpha) = (u, v)$ if $u, v \in (G/N)\rho$~~. It is easily verified (for example, by the criteria of 15.1.1 of [7]) that these define an extension, G^* say, of N^* and that G can be embedded in G^* in such a way that $GN^* = G^*$ and $G \cap N^* = N$. Now N^* is a group with unique extraction of roots, by Theorem 13.6 of Gilbert Baumslag [1], and also $(N^*)'$ is complete, by Theorem 14.5 of Gilbert Baumslag [1]. Hence $(N^*)'$ is isolated in N^* and so, trivially, $(N^*)' \cap N$ is isolated in N. Thus, since $N' \le (N^*)' \cap N$, $I_N(N') \le (N^*)' \cap N$.

In view of 2.2.2 and the fact that N^* has class c, it will suffice to show that $G^*/(N^*)'$ has class at most d. Now $\underline{\underline{N}}_d\{G^*/(N^*)'\}$ is generated by all left normed commutators of weight $d+1$. In fact, using the commutator identities given in an earlier part of this section, it is easy to see that $\underline{\underline{N}}_d\{G^*/(N^*)'\}$ is generated as a normal subgroup of $G^*/(N^*)'$ by all left normed commutators of weight $d+1$ with entries from an arbitrary generating set of $G^*/(N^*)'$ (*cf.* 34.21 of [19]). Since $G^* = GN^*$, $G(N^*)'/(N^*)' \cup N^*/(N^*)'$ is such a generating set and we shall show that all left-normed commutators of weight $d+1$ with entries from this set are trivial.

Now $G(N^*)'/(N^*)' \cong G/G \cap (N^*)'$ and so, since $G \cap (N^*)' \ge N \cap (N^*)' \ge I_N(N')$, $G(N^*)'/(N^*)'$ is nilpotent of class at most d. Thus, if one of our left-normed commutators contains entries from $G(N^*)'/(N^*)'$ only, it will be trivial. Also, since $N^*/(N^*)'$ is an abelian normal subgroup of $G^*/(N^*)'$, if two or more of the entries come from $N^*/(N^*)'$ it will again be trivial. Thus we need only consider commutators of the type $[g_1, \ldots, g_i, n, g_{i+1}, \ldots, g_d]$, where $g_1, \ldots, g_d \in G(N^*)'/(N^*)'$ and $n \in N^*/(N^*)'$.

As N^* is a completion of N, $n^k \in N(N^*)'/(N^*)' \le G(N^*)'/(N^*)'$ for some $k \in \underline{N}$ and so, as we have remarked above,

$$\left[g_1, \ldots, g_i, n^k, g_{i+1}, \ldots, g_d\right] = e .$$

Expanding this commutator by means of our commutator identities, we deduce that

$$[g_1, \ldots, g_i, n, g_{i+1}, \ldots, g_d]^k c = e$$

where c is a product of left-normed commutators each of which involves at least two entries from $N^*/(N^*)'$ and so is trivial. Thus $[g_1, \ldots, g_i, n, g_{i+1}, \ldots, g_d]$ is an element of finite order in the torsion-free group $N^*/(N^*)'$ and so is trivial. Hence all left-normed commutators of weight $d + 1$ with entries from the chosen generating set are trivial and the proof of the theorem follows.

It is well known that a finitely generated nilpotent group has a torsion-free subgroup of finite index. We shall next prove a generalisation of this result which applies to arbitrary nilpotent groups. If G is an arbitrary group, we denote the subgroup generated by all k'th powers $(k \in \underline{N})$ of elements of G by $\underline{\underline{B}}_k(G)$.

2.2.4 LEMMA. *Let* N *be a nilpotent group of class* c *with a torsion subgroup* T *of finite exponent,* m *say. Then, for some natural number* k , $\underline{\underline{B}}_k(N)$ *is torsion-free.*

Proof. Step 1. Suppose firstly that $c \leq 2$ and $N' \leq T$. Let $a, b \in N$. Then, since N has class at most 2 ,

$$(ab)^n = a^n b^n [b, a]^{n(n-1)/2}$$

for any natural number n . Hence, since $[b, a] \in N'$, which has exponent dividing m, the map of N defined by $\tau : a \mapsto a^{2m}$ will be an endomorphism of N with kernel T and image $\underline{\underline{B}}_{2m}(N)$. Hence $\underline{\underline{B}}_{2m}(N) \cong N/T$ which is torsion-free and so the lemma is proved in this case.

Step 2. Suppose now that $N' \leq T$ and N has class c. Step 1 enables us to commence an inductive proof and so we may suppose that $\underline{\underline{B}}_{\ell}\left(N/\underline{\underline{N}}_{c-1}(N)\right)$, that is $\underline{\underline{B}}_{\ell}(N).\underline{\underline{N}}_{c-1}(N)/\underline{\underline{N}}_{c-1}(N)$, is torsion-free for some natural number ℓ. Thus

$$\left(\underline{\underline{B}}_{\ell}(N)\right)' \leq \underline{\underline{B}}_{\ell}(N) \cap N' \leq \underline{\underline{B}}_{\ell}(N) \cap T \leq \underline{\underline{B}}_{\ell}(N) \cap \underline{\underline{N}}_{c-1}(N) \leq \underline{\underline{B}}_{\ell}(N) \cap Z(N) \leq Z\left(\underline{\underline{B}}_{\ell}(N)\right).$$

Hence $\underline{\underline{B}}_{\ell}(N)$ has class at most 2 and $\left(\underline{\underline{B}}_{\ell}(N)\right)' \leq \underline{\underline{B}}_{\ell}(N) \cap T$, the torsion-subgroup of $\underline{\underline{B}}_{\ell}(N)$. We may now apply step 1 to show that, for some $k \in \underline{N}$, $\underline{\underline{B}}_{k}\left(\underline{\underline{B}}_{\ell}(N)\right)$, and so $\underline{\underline{B}}_{k\ell}(N)$, is torsion-free, completing the proof of the lemma in this case.

Step 3. We claim that, if N satisfies the conditions of the lemma, then the torsion subgroup of $N/Z_{c-1}(N)$ has finite exponent dividing m. For, let $a \in N$, $n \in \underline{N}$ and suppose that $a^n \in Z_{c-1}(N)$. Then, for all sequences $b_1, \ldots, b_{c-1}$ of elements of N, $\left[a^n, b_1, \ldots, b_{c-1}\right] = e$. Thus $\left[a, b_1, \ldots, b_{c-1}\right]^n = e$ and so $\left[a, b_1, \ldots, b_{c-1}\right] \in T$. Hence

$$\left[a, b_1, \ldots, b_{c-1}\right]^m = \left[a^m, b_1, \ldots, b_{c-1}\right] = e$$

and so $a^m \in Z_{c-1}(N)$, proving our claim.

Step 4. We are now ready to complete the proof of the lemma by induction on c . The case $c = 1$ was included in step 1. As the torsion subgroup, $T \cap Z_{c-1}(N)$, of $Z_{c-1}(N)$ is certainly of finite exponent, we may apply the induction hypothesis to show that there is a natural number j such that $\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)$ is torsion-free. Now $\underline{\underline{B}}_j\big(Z_{c-1}(N)\big) \trianglelefteq N$ and, since by step 3 the torsion subgroup of $N/Z_{c-1}(N)$ has finite exponent, so also does that of $N/\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)$. Since $N' \leq Z_{c-1}(N)$, $\Big[N/\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)\Big]'$ has finite exponent. Thus we may apply step 2 to show that, for some natural number i , $\underline{\underline{B}}_i\Big[N/\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)\Big]$, that is

$$\underline{\underline{B}}_i(N).\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)/\underline{\underline{B}}_j\big(Z_{c-1}(N)\big) ,$$

is torsion-free. Since $\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)$ is torsion-free, this implies that so is $\underline{\underline{B}}_i(N).\underline{\underline{B}}_j\big(Z_{c-1}(N)\big)$, and therefore also, $\underline{\underline{B}}_i(N)$, completing the proof of the lemma.

We shall say that a group G has max (respectively, max-n) if the subgroups (respectively, the normal subgroups) of G satisfy the maximal condition. Similarly, if $H \trianglelefteq G$, we shall say that H has max-G if the normal subgroups of G contained in H satisfy the maximal condition. It follows that G has max if and only if each subgroup of G is finitely generated and that similar comments hold for max-n and max-G . The following two results, which we offer without proof, are well known and are included here for easier reference. They appear as Lemma 1 and Theorem 3 of P. Hall [8], in a slightly different form.

2.2.5 LEMMA. *Suppose that* G *is a group and that* $H \trianglelefteq G$. *If* G/H *and* H *both have* max *or both have* max-n , *then* G *also has this property. If* G/H *has* max-n *and* H *has* max-G , *then* G *has* max-n .

2.2.6 PROPOSITION. *A finitely generated abelian by nilpotent group has* max-n .

The wreath product construction is of great importance in dealing with product varieties of groups. Since, however, our interest in this construction is not a deep one, we refer to 2.2 of [19] and the references given there for a definition of the wreath product and the terminology and basic results concerning it.

We shall also find occasion to use a lesser known generalisation of the (restricted) wreath product due to B.H. Neumann - the crown product. Let G, H be groups and consider $G \text{ wr } H$. Let K be a subgroup of $Z(G)$. Denote by N the set of all those elements ϕ of the base group, $G^{(H)}$, with the properties

$$\phi(h) \in K \text{ for all } h \in H \text{ , and}$$

$$\prod_{h \in H} \phi(h) = e \text{ .}$$

Then $N \trianglelefteq G \text{ wr } H$ and we define the crown product $G \text{ cr}_K H$ of G and H with amalgamated subgroup K to be $(G \text{ wr } H)/N$. Thus, intuitively, the direct product, $G^{(H)}$, in the wreath product is replaced by the relevant central product in the crown product. We shall be interested only in cases in which $K = Z(G)$ and we shall then write the crown product $G \text{ cr } H$, omitting the subscript K .

Finally, we reserve some notation for groups that we shall use frequently:

$C(n)$ is the cyclic group of order n ,

$C(\infty)$ is the infinite cyclic group,

T_p - with p an odd prime - is the non-abelian group of order p^3 and exponent p ,

T_2 IS THE DIHEDRAL GROUP OF ORDER 8. ~~will denote an arbitrary, but fixed, nonabelian group of order 8~~ .

2.3 Varieties

We refer, once more, to Hanna Neumann's book [19] for basic results concerning varieties of groups. We shall define a variety, after [19], as the class of all groups satisfying each one of a given set of laws. We do, of course, encounter immediate problems as a variety is a proper class. As usual, this problem will be bypassed by noting the one-to-one correspondence, which is described in 14.31 of [19], between varieties and verbal subgroups of an absolutely free group of countably infinite rank. We may then freely apply set-theoretical operations to varieties, without further comment, relying on our ability to transfer these operations to operations on the relevant verbal subgroups of a free group.

We shall denote the word group (of infinite rank) - in the sense of 1.2 of [19] - by X and the distinguished generating set of X (the 'alphabet' of X) by $x_1, x_2, x_3, \ldots$. If $\underline{\underline{V}}$ is a variety, then the (relatively) free group of countably infinite rank (respectively, of finite rank n) will be written $F_\infty(\underline{\underline{V}})$ (respectively, $F_n(\underline{\underline{V}})$) . If G is an arbitrary group, the verbal subgroup of G corresponding to $\underline{\underline{V}}$ will be written $\underline{\underline{V}}(G)$ and the variety generated by G will be written $\operatorname{var} G$. We shall say that $\underline{\underline{V}}$ is torsion-free if, for some torsion-free group G , $\underline{\underline{V}} = \operatorname{var} G$.

Of the commonly used operations on varieties, we refer to 15.82 of [19] for the two basic ones of join ($\vee$) and intersection ($\wedge$) . We recall the definition of the product $\underline{\underline{UV}}$ of two varieties $\underline{\underline{U}}$ and $\underline{\underline{V}}$, as the class of all groups G with a normal subgroup H such that $H \in \underline{\underline{U}}$ and $G/H \in \underline{\underline{V}}$. If $\underline{\underline{U}} = \underline{\underline{V}}$ we shall frequently write $\underline{\underline{UU}} = \underline{\underline{U}}^2$ and similarly for higher powers. We define the commutator $[\underline{\underline{U}}, \underline{\underline{V}}]$ of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ as the class of all those groups G for which $[\underline{\underline{U}}(G), \underline{\underline{V}}(G)] = E$.

We shall now reserve some notation for the following, frequently occuring, varieties:

$\underline{\underline{A}}$ is the variety of all abelian groups,

$\underline{\underline{B}}_n$ is the variety of all groups of finite exponent dividing n ,

$\underline{\underline{A}}_n = \underline{\underline{A}} \wedge \underline{\underline{B}}_n$,

$\underline{\underline{N}}_c$ is the variety of all nilpotent groups of class at most c

(so that $\underline{\underline{N}}_1 = \underline{\underline{A}}$; we shall usually use the latter),

$$\underline{\underline{T}}_p = \operatorname{var} T_p = \begin{cases} \underline{\underline{B}}_p \wedge \underline{\underline{N}}_2 & \text{if } p \text{ is an odd prime,} \\ \underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2 & \text{if } p = 2 \text{ ,} \end{cases}$$

$$\operatorname{var}\left(F_2\left(\underline{\underline{T}}_p\underline{\underline{A}} \wedge [\underline{\underline{E}}, \underline{\underline{A}}^2]\right)\right).$$

$$\underline{\underline{C}}_p = \cancel{\underline{\underline{T}}_p\underline{\underline{A}} \wedge [\underline{\underline{E}}, \underline{\underline{A}}^2] \text{ if } p \text{ is prime.}}$$

2.3.1 LEMMA (*cf.* 2.3 of Peter M. Neumann [21]). *Let* F *be a relatively free group of infinite rank and suppose that* V *is a fully invariant subgroup of* F . *Then* $C_F(V)$ *is fully invariant.*

Proof. Suppose that $\{y_n \mid n \in \underline{N}\}$ is a free generating set of F . Let $c \in C_F(V)$ and let α be an endomorphism of F . Evidently it will be sufficient to show that, if $v \in V$, then $[c\alpha, v] = e$.

Now c may be written using only a finite number of generators of F ; suppose that $c \in \operatorname{gp}(y_1, \ldots, y_n)$. Define the endomorphisms β and γ of F by:

$$\beta : y_i \mapsto y_{i+n} \quad (i \in \underline{N}) \text{ , and}$$

$$\gamma : y_i \mapsto y_i\alpha \quad \text{if } i \leq n$$

$$\gamma : y_i \mapsto y_{i-n} \quad \text{if } i > n \text{ .}$$

We note that $\beta\gamma$ is the identity automorphism and that $c\gamma = c\alpha$. Thus

$$[c\alpha, v] = [c\gamma, v\beta\gamma] = [c, v\beta]\gamma = e$$

as $v\beta \in V$, and the lemma is proved.

We shall need the concept of a verbal Fitting subgroup of a group G. This is defined as the product of all nilpotent verbal subgroups of G. (The parallel with the 'normal' Fitting subgroup is obvious.)

2.3.2 LEMMA. *If F is a soluble, relatively free group of infinite rank and N is its verbal Fitting subgroup, then $C_F(N) \leq N$.*

Proof. The proof is very similar to that of 2.2.1, to which we refer. Firstly, we use 2.3.1 to show that $C_F(N)$ is verbal in F. Then we may repeat the proof of that lemma, replacing 'normal' by 'verbal' throughout (and, of course, G by F).

The following result, quoted here as 2.3.3 and 2.3.4 and due to L.G. Kovács and M.F. Newman, is of great importance in this thesis. Although we shall not need it in the full generality, we quote the result as it appears in 6.1.1 and 6.1.2 of R.A. Bryce [3].

2.3.3 PROPOSITION. *Let $\underline{\underline{V}}$ be a proper subvariety of $\underline{\underline{A}}^2$. Then there exists a unique torsion-free variety $\underline{\underline{T}}$ and a unique natural number n, such that*

$$\underline{\underline{V}} = \underline{\underline{T}} \vee \underline{\underline{A}}_n\underline{\underline{A}} \vee \underline{\underline{P}}$$

where $\underline{\underline{P}}$ has finite exponent.

2.3.4 PROPOSITION. *The varieties* $\underline{\underline{N}}_c\underline{\underline{A}}_s \wedge \underline{\underline{A}}^2$ *are torsion-free and join-irreducible. Every torsion-free proper subvariety of* $\underline{\underline{A}}^2$ *can be uniquely expressed as an irredundant join of some of these torsion-free join irreducibles.*

The following lemma provides an example of how this detailed knowledge of metabelian varieties of exponent zero may be used to give results applying to a wider range of varieties.

2.3.5 LEMMA. *Let* n *and* ℓ *be natural numbers and* $\underline{\underline{V}}$ *a variety such that* $\underline{\underline{A}}_p\underline{\underline{A}} \nleq \underline{\underline{V}}$ *for every prime divisor* p *of* n . *Then there exists a natural number* m *(depending only on* n, ℓ *and* $\underline{\underline{V}}$*) such that, whenever* $H \trianglelefteq G \in \underline{\underline{V}}$ *and* $H \in \underline{\underline{A}}^\ell \wedge \underline{\underline{B}}_n$, $\underline{\underline{B}}_m(G)$ *centralises* H .

Proof. It follows from 2.3.3 that $\underline{\underline{A}}_{\cancel{m}}\underline{\underline{A}} \wedge \underline{\underline{V}} \leq \underline{\underline{A}} \vee \underline{\underline{B}}_k$ for some n natural number k . We use this to prove, by induction on ℓ , that the lemma is valid with $m = k^\ell n^{\ell-1}$. If $\ell = 1$, let H be an abelian normal subgroup, of exponent n , of a group G of $\underline{\underline{V}}$, let $g \in G$, and let $K = gp(H, g)$; then $K \in \underline{\underline{A}}_n\underline{\underline{A}} \wedge \underline{\underline{V}} \leq \underline{\underline{A}} \vee \underline{\underline{B}}_k$ and so K satisfies the common law $[x_1, x_2^k]$ of $\underline{\underline{A}}$ and $\underline{\underline{B}}_k$; in particular, g^k centralises H . Thus in this case $\underline{\underline{B}}_k(G)$ centralises H . For the inductive step, suppose that $\underline{\underline{B}}_m(G)$ centralises H whenever $H \in \underline{\underline{A}}^\ell \wedge \underline{\underline{B}}_n$ and $H \trianglelefteq G \in \underline{\underline{V}}$: we claim that $\underline{\underline{B}}_{kmn}(G)$ centralises H whenever $H \in \underline{\underline{A}}^{\ell+1} \wedge \underline{\underline{B}}_n$ and

$H \trianglelefteq G \in \underline{\underline{V}}$. For, if G and H are as in this claim, then $\underline{\underline{B}}_m(G)$ centralises H' and $\underline{\underline{B}}_k(G/H')$, that is $\underline{\underline{B}}_k(G)H'/H'$, centralises H/H' , so that $\underline{\underline{B}}_{km}(G)$ centralises both H/H' and H' . Now if $h \in H$ and $g \in G$, then $h^{g^{km}} = hh_1$ for some h_1 in H' and $h^{g^{kmn}} = hh_1^n = h$. Thus $\underline{\underline{B}}_{kmn}(G)$ centralises H , as required.

We shall now translate the group theoretical result of 2.2.3 into varietal terms.

2.3.6 LEMMA. *Let* $\underline{\underline{V}}$ *be a variety such that* $\underline{\underline{A}}^2 \nsubseteq \underline{\underline{V}}$, *and let* ℓ *be a natural number. Then there is a bound on the nilpotency class of the torsion-free nilpotent groups of* $\underline{\underline{A}}^\ell \wedge \underline{\underline{V}}$.

Proof. For $\ell = 1$ the claim is trivial. For $\ell = 2$ it follows from 2.3.4: let d be a bound on the class of torsion-free nilpotent groups of $\underline{\underline{A}}^2 \wedge \underline{\underline{V}}$. Suppose, therefore, that every torsion-free nilpotent group in $\underline{\underline{A}}^\ell \wedge \underline{\underline{V}}$ is of class at most c , and let N be a torsion-free nilpotent group in $\underline{\underline{A}}^{\ell+1} \wedge \underline{\underline{V}}$. Since $N' \in \underline{\underline{A}}^\ell \wedge \underline{\underline{V}}$, the class of N' is at most c . Denote $I_{N'}\{\underline{\underline{A}}^2(N)\}$ by I . Then N/I is metabelian and $(N/I)' = N'/I$ is torsion-free. Let T/I be the torsion subgroup of N/I . Then $T \cap N' \leq I$ and N/I is the subdirect product of $N/I/N'/I$ and $N/I/T/I$. However, the former is abelian and the latter is torsion-free and metabelian and therefore has class bounded by d . Thus N/I has class bounded by d . By 2.2.3, N has class bounded by $cd + (c-1)(d-1)$ and the lemma follows.

We now give a number of results, concerning product varieties, which utilise the wreath product construction. We refer for the proof of the first to 17.6 and 22.43 of [19].

2.3.7 LEMMA. *If* $\underline{\underline{V}}$ *is generated by a group* G , *then* $G \text{ wr } C(\infty)$ *generates* $\underline{\underline{VA}}$.

The following lemma was inspired by Lemma 2.2 of Peter M. Neumann [20] and may be regarded as an 'upside-down' version of that result.

2.3.8 LEMMA. *Let* G *be a group which generates the variety* $\underline{\underline{V}}$ *and let* Δ *be an infinite set of natural numbers. Let* $\{H_n \mid n \in \Delta\}$ *be a set of groups with the following property:*

there is a normal subgroup L_n *of* H_n *and an epimorphism* $\lambda_n : L_n \to G$ *and an element* h_n *of* H_n , *such that, if* η_n *denotes the restriction to* L_n *of the inner automorphism induced by* h_n , *then the epimorphisms* $\eta_n^i \lambda_n : L_n \to G \ (0 \le i \le n)$ *are independent in the sense that, whenever* $g_0, \ldots, g_n$ *is a sequence of elements of* G , *there exists an element* ℓ *of* L_n *such that*

$$\ell \eta_n^i \lambda_n = g_i \quad (0 \le i \le n) .$$

Then $\underline{\underline{VA}} \le \text{var}\{H_n \mid n \in \Delta\}$.

(Note: The purpose of this rather unwieldy looking lemma is to deal with the case where L_n has a normal subgroup K_n - the kernel of

λ_n - such that $L_n / \bigcap_{i=0}^{n} K_n^{h_n^i}$ is isomorphic to a direct power of G.

The epimorphisms $\eta_n^i \lambda_n$ give the canonical epimorphisms onto the direct factors and their independence expresses the fact that the above quotient is indeed a direct power.)

Proof. Let $w = w(x_1, \ldots, x_r)$ be a word which is not a law of $\underline{\underline{VA}}$. It will evidently suffice to show, for some $n \in \Delta$, that w is not a law of H_n. Now the independence of the epimorphisms $\eta_n^i \lambda_n$ implies that h_n has order at least $n + 1$. Since Δ is an infinite set, $\underline{\underline{A}} \leq \mathrm{var}\{H_n \mid n \in \Delta\}$ and we may suppose that $w \in X'$.

Let $C = \mathrm{gp}(c)$ be an infinite cyclic group. Then, by 2.3.8, $G \text{ wr } C$ generates $\underline{\underline{VA}}$. Hence w is not a law of $G \text{ wr } C$ and therefore there is a homomorphism $\xi : X \to G \text{ wr } C$ such that $w\xi \neq e$. Suppose that $\xi : x_i \mapsto c^{s(i)}\phi_i$ where $s(i) \in \underline{Z}$ and ϕ_i is an element of the base group of $G \text{ wr } C$. Define $\delta : X \to X$ by

$$\delta : x_i \mapsto x_1^{s(i)} x_{i+1}$$

and $\xi' : X \to G \text{ wr } C$ by

$$\xi' : x_1 \mapsto c \, , \quad x_i \mapsto \phi_{i-1} \quad (i \neq 1) \, .$$

Then $\delta\xi' = \xi$ and so $w\delta\xi' \neq e$. Now, by 22.34 of [19],

$$\begin{aligned} w\delta = w(x_1, \ldots, x_r)\delta &= w\left(x_1^{s(1)} x_2, \ldots, x_1^{s(r)} x_{r+1}\right) \\ &= w\left(x_1^{s(1)}, \ldots, x_1^{s(r)}\right) \prod_{j=1}^{t} x_{i(j)+1}^{\varepsilon(j) x_1^{k(j)}} \end{aligned}$$

where $\varepsilon(j) = \pm 1$, $1 \le i(j) \le r$, $k(j) \in \underline{Z}$. But, since $w \in X'$,

$$w\left(x_1^{s(1)}, \ldots, x_1^{s(r)}\right) = e .$$

Hence

$$w\delta\xi' = \prod_{j=1}^{t} \phi_{i(j)}^{\varepsilon(j)c^{k(j)}} \neq e$$

and so, for some $c^m \in C$,

$$\prod_{j=1}^{t} \phi_{i(j)}^{\varepsilon(j)c^{k(j)}}\left(c^m\right) = \prod_{j=1}^{t} \phi_{i(j)}^{\varepsilon(j)}\left(c^{m-k(j)}\right) \neq e .$$

Denote $-\min_j\left(\{k(j)\}\right)$ by a and $\max_j\left(\{k(j)\}\right)$ by b and let n be an element of Δ such that $n \ge a+b$. We shall show that w is not a law of H_n ; to ease notation we shall drop the subscript n . For each ordered pair of integers u, v satisfying $1 \le u \le r$, $0 \le v \le n$ define $g_{u,v} = \phi_u\left(c^{m+a-v}\right)$. Then, using the independence condition on the $\eta^v\lambda$, there exist, for each u , elements ℓ_u in L such that

$$\ell_u\eta^v\lambda = g_{u,v} = \phi_u\left(c^{m+a-v}\right) .$$

Define $\tau : X \to H$ by

$$\tau : x_1 \mapsto h$$

$$: x_i \mapsto \ell_{i-1} \quad (2 \le i \le r+1)$$

$$: x_i \mapsto e \quad \text{otherwise.}$$

Then

$$w\delta\tau = \left(\prod_{j=1}^{t} x_{i(j)+1}^{\varepsilon(j)x_1^{k(j)}}\right)\tau = \prod_{j=1}^{t} \ell_{i(j)}^{\varepsilon(j)h^{k(j)}} .$$

Hence

$$\begin{aligned} w\delta\tau\eta^{a}\lambda &= \left[\prod_{j=1}^{t} \ell_{i(j)}^{\varepsilon(j)h^{k(j)}}\right]\eta^{a}\lambda \\ &= \prod_{j=1}^{t} \left[\ell_{i(j)}\eta^{a+k(j)}\lambda\right]^{\varepsilon(j)} \\ &= \prod_{j=1}^{t} \phi_{i(j)}^{\varepsilon(j)}\left(c^{m+a-a-k(j)}\right) \\ &= w\delta\xi'_{\Lambda} \neq e . \end{aligned}$$

(c^m)

Thus $w\delta\tau \neq e$ and so w is not a law of H, completing the proof.

For the purposes of the next lemma we make the following convention:

if x, y are elements of the word group X and $f(x) = a_{-m}x^{-m} + \ldots + a_0 + \ldots + a_m x^m$ is an element of $\underline{Z}(gp(x))$, then we define $y^{f(x)}$ to be $\prod_{i=-m}^{m} y^{a_i x^i}$, where, if $i < j$, $y^{a_i x^i}$ occurs before $y^{a_j x^j}$ in the product. As y does not necessarily commute with its conjugates, $\underline{Z}(gp(x))$ is not necessarily a set of operators on X. Whenever we apply the lemma, however, this convention will admit a natural interpretation.

2.3.9 LEMMA. *Let* $f(x_3)$ *be an element of* $\underline{Z}\left[gp(x_3)\right]$ *and let*

$d \in \underline{\underline{A}}^2(X)$. *Then*

i) if $[x_1, x_2]^{f(x_3)} d$ *is a law of* $\underline{\underline{A}}_n\underline{\underline{A}}$, n *is a factor of* $f(x_3)$,

ii) if $[x_1, x_2]^{f(x_3)} d$ *is a law of* $\underline{\underline{A}}^2$, $f(x_3) = 0$.

7 Proof. *i)*. By 2.3.~~8~~, $\underline{\underline{A}}_n\underline{\underline{A}}$ is generated by $C(n)$ wr $C(\infty)$. Suppose that c is a generator of the top group of this wreath product and b is a generator of a coordinate subgroup of the base group. We remark that the base group is then a free $\underline{Z}/(n)(gp(c))$-module which is freely generated by b . Let $\tau : X \to C(n)$ wr $C(\infty)$ be the homomorphism defined by $\tau : x_1 \mapsto b$, $x_2 \mapsto c$, $x_3 \mapsto c$, $x_i \mapsto e$ $(i \neq 1, 2, 3)$. Then $\left([x_1, x_2]^{f(x_3)} d\right)\tau = b^{(c-1)f(c)}$ and therefore, since the above element of X is a law of $\underline{\underline{A}}_n\underline{\underline{A}}$, $b^{(c-1)f(c)} = e$. But, by our remark above, it follows that n is a factor of $(c-1)f(c)$ and so, as is easily seen, a factor of $f(c)$. Thus n is a factor of $f(x_3)$, as required. The proof of *ii)* is similar and we omit it.

Further, we describe a method which will, under certain conditions, provide us with minimal counterexamples. We give this as

2.3.10 LEMMA. *Suppose that* Δ *is a set of finitely based* *varieties and that* $\Omega = \{\underline{\underline{U}} \mid \underline{\underline{U}} \not\geq \underline{\underline{V}}$ *for all* $\underline{\underline{V}} \in \Delta\}$. *Then each element of* Ω *contains a minimal element of* Ω .

Proof. By Zorn's lemma, it is sufficient to show that Ω contains the meet of each descending chain of its elements. Recalling the correspondence between varieties and verbal subgroups of X , we may define sets Δ^* and Ω^* , of verbal subgroups of X , corresponding to Δ and Ω . Then we must show that, if $U \in \Delta^*$, U cannot be contained in the union of any ascending chain of elements of Ω^* . Now U is finitely generated as a verbal subgroup of X and any finite subset of the union of a chain must lie in some term of the chain. Hence, if U were contained in such a union, U would be contained in some term, contradicting the definition of Ω .

2.3.11 LEMMA. *Let* G *be a group with torsion-free derived group. Then* var G *is either abelian or torsion-free.*

Proof. If G is not abelian, it is not of finite exponent and so $\underline{\underline{A}} \leq \text{var } G$. Thus, if we write $V = (\text{var } G)(X)$, $V \leq X'$. Let $u^n \in V$ for some elements u of X and n of $\underline{N}$; then, since X/X' is torsion-free, $u \in X'$. If σ is an arbitrary homomorphism $\sigma : X \to G$, then $\sigma(u^n) = (\sigma(u))^n = e$. But, since $\sigma(u) \in G'$, which is torsion-free, $\sigma(u) = e$. Hence u is a law of G and so $u \in V$. Thus X/V , and so var G , is torsion-free.

CHAPTER 3

VARIETIES NOT CONTAINING ANY $\mathbf{A}\mathbf{A}_p$ OR NOT CONTAINING ANY $\mathbf{A}_p\mathbf{A}$

3.1 Statement of the Theorem

In this chapter we shall prove a number of results involving l.c. the exclusion of the varieties $\mathbf{A}\mathbf{A}_p$ or of the Varieties $\mathbf{A}_p\mathbf{A}$, for p prime. The first form of these results is

3.1.1 THEOREM. *Let* $\mathbf{V}$ *be a soluble variety.*

i) If $\mathbf{A}\mathbf{A}_p \nleq \mathbf{V}$ *for all primes* p *, then there exist natural numbers* n *and* c *such that* $\mathbf{V} \leq \mathbf{B}_n\mathbf{N}_c$.

ii) If $\mathbf{A}_p\mathbf{A} \nleq \mathbf{V}$ *for all primes* p *, then there exist natural numbers* n *and* c *such that* $\mathbf{V} \leq \mathbf{N}_c\mathbf{B}_n$.

iii) If $\mathbf{A}\mathbf{A}_p \nleq \mathbf{V}$ *and* $\mathbf{A}_p\mathbf{A} \nleq \mathbf{V}$ *for all primes* p *, then there exists a nilpotent variety* $\mathbf{N}$ *and a variety* $\mathbf{P}$ *of finite exponent such that* $\mathbf{V} = \mathbf{N} \vee \mathbf{P}$.

The theorem will be proved in Section 3.2. In Section 3.3 we shall introduce a class of varieties - SC-varieties - which is the smallest class of varieties containing all Cross varieties and all soluble varieties which is closed under the operation of taking subvarieties and under the product operation. Then, in Sections 3.4 and 3.5, we shall show that the condition in the statement of 3.1.1, that $\mathbf{V}$ is a soluble variety, may be replaced by the condition that

$\underline{\underline{V}}$ is an SC-variety.

3.2 Proof of Theorem 3.1.1

3.2.1 *Proof of i).* For a proof by contradiction, we suppose that $\underline{\underline{V}}$ is a counterexample to *i)* of 3.1.1. Since $\underline{\underline{V}}$ is soluble, the condition that $\underline{\underline{V}} \le \underline{\underline{B}}_n \underline{\underline{N}}_c$ for some natural numbers n and c is equivalent to the condition that $\underline{\underline{V}} \le \underline{\underline{A}}_k^{\ell} \underline{\underline{N}}_c$ for some natural numbers k, ℓ, and c. However, by 34.14 of [19] and repeated application of Theorem 3.1 of Graham Higman [13], every variety in the set $\left\{\underline{\underline{A}}_k^{\ell} \underline{\underline{N}}_c \mid k, \ell, c \in \underline{N}\right\}$ is finitely based. Thus an application of 2.3.10 shows that $\underline{\underline{V}}$ contains a subvariety which is a minimal counterexample. We may suppose without loss of generality that $\underline{\underline{V}}$ is this minimal counterexample.

Denote $F_\infty(\underline{\underline{V}})$ by F and let T be a verbal torsion subgroup of F. Since T is soluble, if it were non-trivial, there would be a non-trivial abelian subgroup A of T, verbal in T and so verbal in F. Also A would have a fully invariant subgroup, B, of finite exponent and so B would be verbal in F (for example, since A must have a non-trivial p-element for some prime p, we could take B as the subgroup of all elements of order dividing p). Then F/B would generate a proper subvariety of $\underline{\underline{V}}$ and it is immediate that $\underline{\underline{V}}$ could not be a minimal counterexample in this case. Thus T is trivial. In particular, every nilpotent verbal subgroup of F is torsion-free. Since the conditions of the theorem imply that $\underline{\underline{A}}^2 \nleq \underline{\underline{V}}$, 2.3.6 shows that $\underline{\underline{V}}$ has a bound on the class of its torsion-free nilpotent groups. Thus the verbal Fitting subgroup of F, N say, is nilpotent and torsion-free.

Suppose that F/N is non-trivial. Then there is a verbal subgroup, A say, of F such that A/N is abelian and non-trivial. Since $A' \leq N$, $A/I_N(N')$ is a metabelian group with torsion-free derived group. By 2.3.11, $\operatorname{var}\{A/I_N(N')\}$ is either abelian or torsion-free. The conditions of the theorem, together with 2.3.4, however, show that the torsion-free metabelian subvarieties of $\underline{\underline{V}}$ are nilpotent. Hence, in either case, $A/I_N(N')$ is nilpotent. Thus, by 2.2.3, A is nilpotent. Then, since A is verbal in F, $A \leq N$, contradicting the assumption that A/N is non-trivial. So F/N is trivial and $\underline{\underline{V}}$ is nilpotent and so certainly not a counterexample as we supposed. The proof of *i)* is complete.

3.2.2 *Proof of ii)*. The proof is again by contradiction. Since, by Theorem 3.1 of Graham Higman [13] all varieties of the set $\{\underline{\underline{N}}_c\underline{\underline{B}}_n \mid c, n \in \underline{N}\}$ are finitely based, we may, as in 3.2.1, suppose that $\underline{\underline{V}}$ is a minimal counterexample to *ii)* of 3.1.1. Denote $F_\infty(\underline{\underline{V}})$ by F. Then the proof splits into two cases:

a) F has a non-trivial, verbal, torsion subgroup.

In this case, as in 3.2.1, F has a non-trivial, elementary abelian, verbal subgroup, A say. Then F/A generates a proper subvariety of $\underline{\underline{V}}$ and so $F/A \in \underline{\underline{N}}_c\underline{\underline{B}}_n$ for some natural numbers c, n. Since $\underline{\underline{A}}_p\underline{\underline{A}} \nsubseteq \underline{\underline{V}}$ for all primes p, 2.3.5 shows that $\underline{\underline{B}}_m(F)$ centralises A for some natural number m. Now

$$\left(\underline{\underline{B}}_m(F).A \cap \underline{\underline{B}}_n(F).A\right)/A \le \underline{\underline{B}}_n(F).A/A = \underline{\underline{B}}_n(F/A) ,$$

which is nilpotent. Thus, since A is central in $\underline{\underline{B}}_m(F).A$, and so in $\left(\underline{\underline{B}}_m(F).A \cap \underline{\underline{B}}_n(F).A\right)$, the latter subgroup is also nilpotent. Since $F/\left(\underline{\underline{B}}_m(F).A \cap \underline{\underline{B}}_n(F).A\right)$ is of finite exponent, $\underline{\underline{V}}$ cannot be a counterexample as we supposed, which completes the proof in this case.

b) F has no non-trivial, verbal, torsion subgroup.

In this case, as in 3.2.1, the verbal Fitting subgroup, N say, of F is nilpotent and torsion-free. Since F is soluble N is non-trivial and so F/N generates a proper subvariety of $\underline{\underline{V}}$; suppose that $F/N \in \underline{\underline{N}}_c\underline{\underline{B}}_n$ $(c, n \in \underline{N})$. Let A be the subgroup of F defined by $A/N = Z\left(\underline{\underline{B}}_n(F/N)\right)$. Then, by 2.3.1, A is verbal in F and since $\underline{\underline{B}}_n(F/N)$ is nilpotent, A/N is non-trivial. Let I denote $I_N(N')$. Then A/I is metabelian and the derived group of A/I is torsion-free. Hence, by 2.3.11 and 2.3.4, $A/I \in \underline{\underline{N}}_d\underline{\underline{B}}_m$ for some natural numbers d, m. Put $\underline{\underline{B}}_m(A/I) = B/I$. Then B is verbal in F and B/I, and so BN/I, is nilpotent. Hence, by 2.2.3, BN is nilpotent. Thus $B \le N$ and so $A/N \in \underline{\underline{B}}_m$. But A/N was defined as $Z\left(\underline{\underline{B}}_n(F/N)\right)$, so by Corollary 1.62 of [25], $\underline{\underline{B}}_n(F/N)$ has finite exponent. Thus F/N has finite exponent, $\underline{\underline{V}}$ cannot be a counterexample in this case either, and the proof is complete.

In the light of parts *i)* and *ii)*, we restate *iii)* in a form that will be more useful later.

3.2.3 LEMMA. *Let* $\mathbf{V}$ *be a variety (not necessarily soluble) and suppose that* $\mathbf{V} \leq \mathbf{B}_n\mathbf{N}_c \wedge \mathbf{N}_c\mathbf{B}_n$ *for some natural numbers* n, c . *Then for some nilpotent variety* $\mathbf{N}$ *and variety* $\mathbf{P}$ *of finite exponent,* $\mathbf{V} = \mathbf{N} \vee \mathbf{P}$.

Proof. Denote $F_\infty(\mathbf{V})$ by F , $\mathbf{B}_n(F)$ by B , $\mathbf{N}_c(F)$ by N , and the torsion subgroup of the nilpotent group B by T . Since $B \cap N \in \mathbf{B}_n$, $T \geq B \cap N$. Thus $T/B \cap N = T/T \cap N \cong TN/N$. But F/N , as a nilpotent free group, has maximum condition on its fully
19 invariant subgroups (this follows from 34.14 of [$\cancel{1}$]); in particular, the torsion subgroup of F/N has finite exponent. Thus $TN/N \cong T/B \cap N$ has finite exponent and so T has finite exponent. Hence, by 2.2.4, $\mathbf{B}_m(B) \cap T = E$ for some natural number m . Therefore, $\mathbf{B}_m(B) \cap N = \mathbf{B}_m(B) \cap (B \cap N) \leq \mathbf{B}_m(B) \cap T = E$. Since $\mathbf{B}_m(B) \geq \mathbf{B}_{mn}(F)$ and $N = \mathbf{N}_c(F)$, $\mathbf{V} = (\mathbf{B}_{mn} \wedge \mathbf{V}) \vee (\mathbf{N}_c \wedge \mathbf{V})$ and the proof is complete.

3.3 SC-varieties

3.3.1 DEFINITION. A variety $\mathbf{V}$ is called an SC-*variety* if there are varieties $\mathbf{S}_1, \ldots, \mathbf{S}_n$ such that,

$$\mathbf{V} \leq \mathbf{S}_1 \ \ldots \ \mathbf{S}_n$$

and, for each i , either

a) $\mathbf{S}_i$ is Cross, or

b) $\underline{\underline{S}}_i$ is soluble.

NOTE. It follows immediately that the conditions above may be replaced by

a') $\underline{\underline{S}}_i$ is generated by a finite simple group,

b') $\underline{\underline{S}}_i = \underline{\underline{A}}$.

Thus the class of SC-varieties is the smallest class of varieties which contains all Cross varieties and all soluble varieties and which is closed under the product operation and the operation of taking subvarieties.

The following theorem will be proved.

3.3.2 THEOREM. *Let* $\underline{\underline{V}}$ *be an* SC-*variety. Then i), ii) and iii) of* 3.1.1 *hold for* $\underline{\underline{V}}$.

Proof. Since $\underline{\underline{V}}$ is an SC-variety, there are varieties $\underline{\underline{S}}_1, \ldots, \underline{\underline{S}}_k$ each of which is either abelian or Cross such that $\underline{\underline{V}} \leq \underline{\underline{S}}_1 \ldots \underline{\underline{S}}_k$ and such that k is the least natural number for which such an expression exists.

We commence the proof of 3.3.2 by induction on k . If $k = 1$ the result is trivial in any of the three cases. Consider part *i)*. We may suppose that $\underline{\underline{V}} \wedge (\underline{\underline{S}}_2 \ldots \underline{\underline{S}}_k) \leq \underline{\underline{B}}_n\underline{\underline{N}}_c$ for some natural numbers n, c . If $\underline{\underline{S}}_1$ is Cross, the result follows immediately. Otherwise $\underline{\underline{S}}_1$ is abelian and $\underline{\underline{V}} \leq \underline{\underline{A}}\underline{\underline{B}}_n\underline{\underline{N}}_c$. If we could prove part *i)* for

$\mathbf{V} \wedge \mathbf{AB}_n$ then it would follow that $\mathbf{V} \leq \mathbf{B}_m\mathbf{N}_d\mathbf{N}_c$ for some natural numbers m, d. Since, by part *i)* of 3.1.1, the theorem is true for $\mathbf{V} \wedge \mathbf{N}_d\mathbf{N}_c$, it would then also be true for $\mathbf{V}$. Hence we see that it is sufficient to prove

3.3.3 LEMMA. *Let* $\mathbf{V} \leq \mathbf{AB}_m$ $(m \in \mathbf{N})$ *be an* SC-*variety and suppose that* $\mathbf{AA}_p \nleq \mathbf{V}$ *for all primes* p. *Then* $\mathbf{V} \leq \mathbf{B}_n\mathbf{N}_c$ *for some natural numbers* n, c.

Repeating the induction for the proof of part *ii)*, we may suppose that $\mathbf{V} \wedge (\mathbf{S}_2 \ldots \mathbf{S}_k) \leq \mathbf{N}_c\mathbf{B}_n$ for some natural numbers c, n. If $\mathbf{S}_1$ is abelian, then $\mathbf{V} \leq \mathbf{AN}_c\mathbf{B}_m$ and we may apply part *ii)* of 3.1.1 to show that $\mathbf{V} \wedge \mathbf{AN}_c$ satisfies the conclusion of the theorem and so $\mathbf{V}$ satisfies the conclusion of the theorem. Hence we need only consider the case where $\mathbf{S}_1$ is a Cross variety, and so it will be sufficient to prove

3.3.4 LEMMA. *Let* $\mathbf{V} \leq \mathbf{SN}_c$ $(c \in \mathbf{N})$ *where* $\mathbf{S}$ *is a Cross variety and suppose that* $\mathbf{A}_p\mathbf{A} \nleq \mathbf{V}$ *for all primes* p. *Then* $\mathbf{V} \leq \mathbf{N}_d\mathbf{B}_n$ *for some natural numbers* d, n.

The proofs of these two lemmas will be given in the next two sections. We would note here that the methods used in the proof of 3.3.4 are taken from some joint work, done in a different context, with L.G. Kovács. In particular, the ideas behind the use of 2.3.8 are his.

For the proof of 3.3.3, we shall also need the following lemma - a varietal analogue to Theorem 5.3 of B.H. Neumann [17].

3.3.5 LEMMA. *If* $\mathbf{P}$ *is a locally finite variety, then* $[\mathbf{E}, \mathbf{P}] \leq \mathbf{B}_n\mathbf{A}$ *for some natural number* n .

Proof. Let the exponent of $\mathbf{P}$ be m . By a theorem of
2 Roger M. Bryant [~~17~~], $\mathbf{B}_m \vee \mathbf{A}$ is finitely based, suppose that $w = w(x_1, \ldots, x_k)$ is a basis for its laws. Denote $F_k([\mathbf{E}, \mathbf{P}])$ by F and let $y_1, \ldots, y_k$ be a free generating set of F . Then $F/\mathbf{P}(F)$ is a finitely generated group of $\mathbf{P}$ and so is finite. Hence $\mathbf{P}(F)$ is a central subgroup of finite index in F and so, by Theorem 5.3 of [17], F' is finite. In particular, $\mathbf{B}_m(F) \cap F'$ is of finite exponent, ℓ say. Thus, since $w(y_1, \ldots, y_k) \in \mathbf{B}_m(F) \cap F'$, $w(y_1, \ldots, y_k)^\ell = e$.

Denote $F_\infty([\mathbf{E}, \mathbf{P}])$ by H and let $y_1, \ldots, y_n, \ldots$ be a free generating set of H . Then the relation $w(y_1, \ldots, y_k)^\ell = e$ holds in H also. But the word w is a basis for the laws of the variety $\mathbf{A} \vee \mathbf{B}_m$ and so $(\mathbf{A} \vee \mathbf{B}_m)(H) \leq H' \cap \mathbf{B}_m(H)$ is generated by all endomorphic images of $w(y_1, \ldots, y_k)$. Evidently each of these images is of order dividing ℓ . Since $H' \cap \mathbf{B}_m(H)$ is abelian, even central, $(\mathbf{A} \vee \mathbf{B}_m)(H)$ is thus of finite exponent dividing ℓ . Therefore $H \in \mathbf{B}_\ell(\mathbf{A} \vee \mathbf{B}_m) \leq \mathbf{B}_\ell\mathbf{B}_m\mathbf{A} \leq \mathbf{B}_{\ell m}\mathbf{A}$ and so $[\mathbf{E}, \mathbf{P}] \leq \mathbf{B}_n\mathbf{A}$ where $n = \ell m$.

3.4 Proof of Lemma 3.3.3

The proof is by contradiction; we suppose that $\underline{\underline{V}}$ is a counterexample to the lemma. By 3.2.3, a subvariety of $\underline{\underline{V}}$ satisfies the conclusion of the lemma if and only if it is a subvariety of one of the varieties of the set $\{\underline{\underline{N}}_c \vee \underline{\underline{B}}_n \mid c, n \in \underline{N}\}$. Since, by a theorem of Roger M. Bryant [2], every variety of this set is finitely
10 based, we may apply 2.3.~~1~~1 and suppose that $\underline{\underline{V}}$ is a minimal counterexample.

Let F denote $F_\infty(\underline{\underline{V}})$ and B denote $\underline{\underline{B}}_m(F)$. Now the torsion subgroup of B is fully invariant in B and so verbal in F; if it were non-trivial, it would give rise to a non-trivial elementary abelian verbal subgroup of F. In this case, as in 3.2.1, $\underline{\underline{V}}$ could not be a minimal counterexample. Hence B is torsion-free. Let $x \in F$. Then, if $H = gp(B, x)$, $H \in \underline{\underline{AA}}_m$ and $H' \leq B$ is torsion-free. Thus by 2.3.11, var H is a torsion-free subvariety of $\underline{\underline{AA}}_m$; now the conditions of the lemma and 2.3.4 show that var $H = \underline{\underline{A}}$. This shows that every element of F commutes with B; that is, $B \leq Z(F)$ and so $\underline{\underline{V}} \leq [\underline{\underline{E}}, \underline{\underline{V}} \wedge \underline{\underline{B}}_m]$. Since $\underline{\underline{V}} \wedge \underline{\underline{B}}_m$ is an SC-variety of finite exponent, it is locally finite and so 3.3.5 shows that $\underline{\underline{V}} \leq \underline{\underline{B}}_n\underline{\underline{A}}$ for some natural number n, contradicting the choice of $\underline{\underline{V}}$ as a counterexample and completing the proof of the lemma.

3.5 Proof of Lemma 3.3.4

The proof is by contradiction. Using a method similar to that in 3.4, we may suppose that $\underline{\underline{V}}$ is a minimal counterexample to the lemma. Denote $F_\infty(\underline{\underline{V}})$ by F and let y be a free generator of F.

Since $F \in \underline{\underline{SN}}_c$ and the Cross variety $\underline{\underline{S}}$ has only finitely many subvarieties, $\underline{\underline{N}}_c(F)$ has a minimal verbal subgroup, W say; evidently, W is verbally simple and is verbal in F. Since $W \in \underline{\underline{S}}$ and subvarieties of Cross varieties are Cross, var W can be generated by a finite group. Let G be a finite group of least order generating var W.

We claim that G is simple. For, let $N \trianglelefteq G$. Then $W \in \text{var } G \leq (\text{var } N)\big(\text{var}(G/N)\big)$. But, since W is verbally simple, the verbal subgroup of W corresponding to $\text{var}(G/N)$ is either trivial or W itself. In the first case, $W \in \text{var}(G/N)$ and so $N = E$, by the minimality of G; in the second case, $W \in \text{var } N$ and so $N = G$, again by the minimality of G. Thus G is simple. If G is abelian, then $\underline{\underline{V}} \leq \underline{\underline{A}}\ \text{var}(F/W) \leq \underline{\underline{AN}}_d\underline{\underline{B}}_m$ for some natural numbers d and m, as F/W generates a proper subvariety of $\underline{\underline{V}}$, and by 3.1.1 this contradicts the choice of $\underline{\underline{V}}$ as a minimal counterexample. Hence G is a non-abelian finite simple group. Denote the subvariety of var G generated by the proper sections of G by $\underline{\underline{R}}$.

Let Γ be the set $\{N \mid N \trianglelefteq W \text{ and } W/N \cong G\}$.

We claim that $\bigcap_{N \in \Gamma} N = E$. Evidently, it will suffice to show that, if $e \neq w \in W$, then there is a normal subgroup N of W such that

$w \notin N$ and $W/N \cong G$. With this in mind, let N be a normal subgroup of G maximal with respect to avoiding w and let M denote $\mathrm{nsgp}_W(w, N)$; the existence of N is guaranteed by Zorn's lemma. Then M/N is a chief factor of W and W/N is monolithic. Also, since W lies in the Cross variety $\underline{\underline{S}}$, 52.21 of [19] shows that M/N is finite. In order to show that $W/N \cong G$, we must firstly show that M/N is non-abelian.

Since $\mathrm{var}\, W = \mathrm{var}\, G$ and W is verbally simple, $\underline{\underline{R}}(W) = W$. Now $\underline{\underline{R}}$, being a Cross variety, is finitely based (see, for example, 52.12 of [19]); let the word $r(x_1, \ldots, x_k)$ be a basis for the laws of $\underline{\underline{R}}$. Since $w \in W = \underline{\underline{R}}(W)$, there are elements a_{ij} $(1 \le i \le k,\ 1 \le j \le \ell)$ of W such that

$$w = \prod_{1\le j\le \ell} r(a_{1j}, \ldots, a_{kj}) .$$

Let $Y = \mathrm{gp}(\{a_{ij} \mid 1 \le i \le k,\ 1 \le j \le \ell\}, M)$. Then Y/M is a finitely generated group of the Cross variety $\underline{\underline{S}}$ and therefore it is finite. Since M/N is also finite, Y/N is finite. Now $wN \in \underline{\underline{R}}(Y/N) \cap M/N$ and wN is non-trivial. Thus $\underline{\underline{R}}(Y/N) \cap M/N$ is non-trivial. As Peter M. Neumann noted ([21], p. 77), the proof of Lemma 3.2 of Sheila Oates [23] shows that the $\underline{\underline{R}}$-subgroups of the finite groups in $\mathrm{var}\, G$ are direct powers of G (the additional hypotheses of Oates being unnecessary). Hence $\underline{\underline{R}}(Y/N) \cap M/N$, being a normal subgroup of $\underline{\underline{R}}(Y/N)$, is also such a direct power. In particular, M/N is non-abelian. By Theorem 4 of L.G. Kovács and M.F. Newman [14], W/N is isomorphic to a section of G . Since $\underline{\underline{R}}(W/N) = \underline{\underline{R}}(W)N/N = W/N$ is non-trivial, this section cannot be a group of $\underline{\underline{R}}$. Hence $W/N \cong G$, as required.

The proof now splits into two cases. Suppose, firstly, that for some $n \in \underline{N}$, $N^{y^n} = N$ for all $N \in \Gamma$. Then y^n normalises N and, if m is the exponent of $\operatorname{Aut} G$, y^{nm} centralises W/N , for all $N \in \Gamma$. Hence $[W, y^{nm}] \leq \bigcap_{N \in \Gamma} N = E$ and so $y^{nm} \in C_F(W)$. But, by 2.3.1, $C_F(W)$ is verbal in F and so, since y is a free generator of F , $\underline{\underline{B}}_{mn}(F) \leq C_F(W)$. Now $\underline{\underline{B}}_{mn}(F)W/W$ is nilpotent by finite exponent and therefore so also is $\underline{\underline{B}}_{mn}(F)/(\underline{\underline{B}}_{mn}(F) \cap W)$. But we have shown that $\underline{\underline{B}}_{mn}(F) \cap W$ is central in $\underline{\underline{B}}_{mn}(F)$. Thus $\underline{\underline{B}}_{mn}(F)$, and so F itself, is nilpotent by finite exponent, which contradicts the choice of $\underline{V}$ as a minimal counterexample.

Thus we may reject the hypothesis in the first case and suppose that, for all $n \in \underline{N}$, there is an $N_n \in \Gamma$ such that $N_n^{y^{n!}} \neq N_n$. In preparation for an application of 2.3.8, we will show that $W/\bigcap_{0 \leq i \leq n} N_n^{y^i} \cong G^{n+1}$ (that is, the direct product of $n+1$ copies of G). Since $N_n^{y^{n!}} \neq N_n$, $N_n^{y^i} \neq N_n^{y^j}$ whenever $0 \leq i < j \leq n$. Now, as $\underline{\underline{R}}\left(W/\bigcap_{0 \leq i \leq n} N_n^{y^i}\right) = W/\bigcap_{0 \leq i \leq n} N_n^{y^i}$ and $\underline{\underline{R}}$-subgroups of FINITE groups in $\operatorname{var} G$ are direct powers of G , $W/\bigcap_{0 \leq i \leq n} N_n^{y^i}$ is such a direct power. Also, $W/\bigcap_{0 \leq i \leq n} N_n^{y^i}$ evidently has order at most $|G|^{n+1}$ and has at least $n+1$ distinct maximal normal subgroups. Thus $W/\bigcap_{0\le i\le n} N_n^{y^i} \cong G^{n+1}$, as required.

Transferring from the notation of 2.3.8, we let $\Delta = \underline{N}$, $H_n = F$, $h_n = y^{-1}$, $L_n = W$, and let δ_n be the homomorphism $\delta_n : W \to G$ with kernel N_n. Then η_n is the restriction to W of the inner automorphism of F induced by y^{-1} and the kernel of $\eta_n^i\delta_n$ is $N_n^{y^i}$ $(0 \le i \le n)$. Let $\pi_0, \ldots, \pi_i, \ldots, \pi_n$ be the canonical projections $\pi_i : G^{n+1} \to G$ which define G^{n+1}. It is a defining property of (cartesian) products that, given the homomorphisms $\delta_n, \eta_n\delta_n, \ldots, \eta_n^n\delta_n : W \to G$, there is a unique homomorphism $\theta : W \to G^{n+1}$ such that $\theta\pi_i = \eta_n^i\delta_n$ $(0 \le i \le n)$ and $\ker\theta = \bigcap_{0\le i\le n} \ker \eta_n^i\delta_n$. Then

$$|W\theta| = \left|W/\bigcap_{0\le i\le n} \ker \eta_n^i\delta_n\right| = \left|W/\bigcap_{0\le i\le n} N_n^{y^i}\right| = |G^{n+1}|$$

and therefore, as G^{n+1} is finite, $W\theta \cong G^{n+1}$. Thus, if $g_0, \ldots, g_n \in G$, we may take an element h of G^{n+1} such that

$$h\pi_i = g_i \quad (0 \le i \le n)$$

and then an element v of W such that $v\theta = h$, so that

$$v\eta_n^i\delta_n = v\theta\pi_i = g_i \quad (0 \le i \le n) .$$

Hence the requirements of 2.3.8 are satisfied and so $(\mathrm{var}\ G)\underline{A} \in \underline{V}$. But then, if p is a prime divisor of $|G|$, $\underline{A}_p\underline{A} \le \underline{V}$, contradicting the hypothesis of the lemma and completing the proof.

CHAPTER 4

VARIETIES NOT CONTAINING $\underline{\underline{A}}^2$

4.1 Introduction

In this chapter we shall prove the following result:

4.1.1 THEOREM. *Let* $\underline{\underline{V}}$ *be a soluble variety such that*

i) all subvarieties of $\underline{\underline{V}} \wedge \underline{\underline{A}}\underline{\underline{N}}_2\underline{\underline{A}}$ *can be generated by the finite groups they contain.*

Then, if $\underline{\underline{V}}$ *does not contain* $\underline{\underline{A}}^2$,

ii) there exist natural numbers n, c *such that* $\underline{\underline{V}} \leq \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$.

The proof will take up the remainder of the chapter. In Section 2, we shall show that we may suppose that $\underline{\underline{V}}$ is a minimal counterexample to the theorem and that $\underline{\underline{V}} \leq \underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$, where p is prime. We shall also show that the theorem holds for metanilpotent varieties; the condition *i)* being unnecessary in this case. In Section 3, we shall investigate the properties of $\underline{\underline{V}}$ and, in particular, the residual properties of a free group of $\underline{\underline{V}}$, to obtain the required contradiction.

4.2 Development of a minimal counterexample

Suppose that $\underline{\underline{V}}$ is a counterexample to the theorem. Now if a subvariety, $\underline{\underline{W}}$, of $\underline{\underline{V}}$ satisfies *ii)* of 4.1.1, we may equally well suppose, since $\underline{\underline{W}}$ is soluble, that $\underline{\underline{W}} \leq \underline{\underline{A}}_k^{\ell}\underline{\underline{N}}_c\underline{\underline{A}}_k^{\ell}$ for some natural numbers k, ℓ and c. However, by repeated use of Theorem 3.1 of Graham Higman [13], each of the latter varieties is finitely based. Thus, applying 2.3.10, $\underline{\underline{V}}$ contains minimal counterexamples to the theorem and we may as well suppose that $\underline{\underline{V}}$ is one of these minimal counterexamples.

by

Denote $F_\infty(\underline{\underline{V}})$ ~~be~~ G and let $y_1, \ldots, y_k, \ldots$ be a free generating set of G. Let T be a verbal torsion subgroup of G. If T is non-trivial, then, since it is soluble, there is a non-trivial abelian subgroup of T, fully invariant in T and so verbal in F. This abelian subgroup, in turn, must have a non-trivial subgroup of finite exponent which is verbal in F. But this evidently contradicts the choice of $\underline{\underline{V}}$ as a minimal counterexample and so T = E. In particular, all nilpotent verbal subgroups of F are torsion-free and so, by 2.3.6, there is a bound on their class. Thus, if we denote the verbal Fitting subgroup of G by N, N is nilpotent and torsion-free. We also note that, as $\underline{\underline{V}}$ is a minimal counterexample to the theorem, $\underline{\underline{B}}_n(G)$ generates $\underline{\underline{V}}$ for each natural number n.

Before proceeding with the main part of the proof, we shall, as promised, prove the theorem in the case that $\underline{\underline{V}}$ is metanilpotent.

4.2.1 PROPOSITION. *Theorem* 4.1.1 *is true, regardless of i), for metanilpotent varieties.*

Proof. Since G is metanilpotent, G/N is nilpotent. Let

$Z \leq G$ be such that Z/N is the centre of G/N. Then, by 2.3.1, Z is verbal in G. Denote $I_N(N')$ by I; then I is verbal in G. Now Z/I is a metabelian group and, since $Z' \leq N$, the derived group $Z'I/I$ is torsion-free. It follows, from the descriptions in 2.3.3 and 2.3.4 of metabelian varieties of exponent zero, that $Z/I \in \underline{\underline{N}}_c\underline{\underline{A}}_s$ for some natural numbers c and s.

Let $M \leq G$ be such that $M/I = \underline{\underline{A}}_s(Z/I)$. Then M is verbal in G and M/I is nilpotent. Thus, since N/I is nilpotent, MN/I is nilpotent. It follows, by 2.2.3, that MN is nilpotent. Since MN is also verbal in F, it is contained in the verbal Fitting subgroup N. Thus $M \leq N$.

Hence $Z/N \in \underline{\underline{A}}_s$ and so, since the centre of G/N has finite exponent, G/N does also (see, for example, Lemma 1.62 of Derek J. S. Robinson [25]). Thus G is nilpotent by finite exponent and so $\underline{\underline{V}}$ cannot be a counterexample. The proof is complete.

We shall now show that $\underline{\underline{V}} \leq \underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ for some prime p and natural number t. We shall accomplish this in a number of steps.

4.2.A. *There exist natural numbers* n, c *such that* $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{B}}_n\underline{\underline{A}}$.

Proof. G' certainly generates a proper subvariety of $\underline{\underline{V}}$ since it has lesser solubility length than G. Thus $\operatorname{var} G'$ satisfies *ii)* of 4.1.1. Also, any torsion verbal subgroup of G' is a torsion verbal subgroup of G and so is trivial. Thus G' is nilpotent by finite exponent and the result follows.

4.2.B. $\underline{\underline{V}} \leq \underline{\underline{N}}_c(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_2)\underline{\underline{A}}$.

Proof. Firstly we note that $N \leq G'$; for otherwise $\mathrm{var}(G/G') = \underline{\underline{A}} \nleq \mathrm{var}(G/N)$ and so the latter has finite exponent. But then G is nilpotent by finite exponent and $\underline{\underline{V}}$ cannot be a counterexample as we supposed.

We have shown in 4.2.A that $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{B}}_n\underline{\underline{A}}$ and so G'/N is soluble of finite exponent. Thus there is a series

$$N = B_1 < B_2 < \ldots < B_{k-1} < B_k = G'$$

where

1) each B_i is verbal in G ,

2) B_i/B_{i-1} is elementary abelian, of exponent $p(i)$ say,

3) k is the least natural number for which such a series exists.

If $k \leq 2$, then 4.2.B is immediately true and so we may suppose that $k > 2$. Also, if $x \in G$ and $gp(B_{k-1}, x) = H$, say, then H does not generate $\underline{\underline{V}}$ because 3) implies that

$$\underline{\underline{V}} \nleq (\mathrm{var}\ N)\underline{\underline{A}}_{p(2)}\underline{\underline{A}}_{p(3)} \cdots \underline{\underline{A}}_{p(k-1)}\underline{\underline{A}} .$$

Thus var H satisfies *ii)* of 4.1.1. Suppose, however, that T is a torsion verbal subgroup of H . Then $T \cap N = E$ and so $[T, N] = E$. But N contains its centraliser in G and is torsion-free, and so $T = E$. Thus H is nilpotent by finite exponent and so $\underline{\underline{A}}_{p(i)}\underline{\underline{A}} \nsubseteq \mathrm{var}(H)$ $(1 \le i \le k-1)$. We may now apply 2.3.5 to show that, for some natural number m , x^m centralises B_{k-1}/N and so, since x was arbitrary in G , $\underline{\underline{B}}_m(G)$ centralises B_{k-1}/N .

Hence $\left(\underline{\underline{B}}_m(G)N \cap G'\right)/N$ is nilpotent of class at most 2 and,

N of course, $\underline{\underline{B}}_m(G)/\left(\underline{\underline{B}}_m(G)N \cap G'\right)$ is abelian. But $\underline{\underline{B}}_m(G)N$, even $\underline{\underline{B}}_m(G)$, generates $\underline{\underline{V}}$ and we have proved 4.2.B.

4.2.C. $\underline{\underline{V}} \le \underline{\underline{N}}_c\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ *for some prime power divisor* p^t *of* n .

Proof. Owing to 4.2.B, G'/N is nilpotent of finite exponent and therefore it is the direct product of its Sylow subgroups - say

$$G'/N = P_1/N \times \ldots \times P_k/N .$$

Our claim amounts to $k = 1$; suppose that this is not the case. Then, if $x \in G$, $K_i = \mathrm{gp}\left(P_i, x\right)$ generates a proper subvariety of $\underline{\underline{V}}$.

Using a method similar to that used in the proof of 4.2.B, we show that, for each i $(1 \le i \le k)$, $\underline{\underline{B}}_{m(i)}$ centralises P_i/N for some natural number $m(i)$. Then, if m is a common multiple of $m(1), \ldots, m(k)$, $\underline{\underline{B}}_m(G)$ centralises P_i/N for all i and so centralises G'/N .

Thus $\underline{\underline{B}}_m(G)N/N$ is nilpotent of class at most 2 and so $\underline{\underline{B}}_m(G)N$ is metanilpotent. Since $\underline{\underline{B}}_m(G)N$ generates $\underline{\underline{V}}$, 4.2.1 shows that $\underline{\underline{V}}$ cannot be a counterexample. Hence $k > 1$ yields a contradiction and 4.2.C is proved.

4.2.D. $\underline{\underline{V}} \leq \underline{\underline{A}}\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$.

Proof. In the light of 4.2.C it will evidently suffice to prove that N is abelian. Suppose not. Then, if we denote $I_N(N')$ by I, I is non-trivial. Since I is verbal, G/I generates a proper subvariety of $\underline{\underline{V}}$ which thus satisfies *ii)* of 4.1.1. Also, an application of 2.2.3 shows that N/I is the verbal Fitting subgroup of G/I and so, by 2.3.2, is self-centralising. Hence, if T/I is a verbal torsion subgroup of G/I, $T \cap N \leq I$ - since N/I is torsion-free - and so $[T, N] \leq I$. Thus T/I is trivial. Hence G/I is nilpotent by finite exponent and G is metanilpotent by finite exponent. We may now use 4.2.1 to show that $\underline{\underline{V}}$ is not a counterexample, contrary to our supposition. Therefore N is abelian and 4.2.D is proved.

In the remaining section we shall abbreviate $\left(\underline{\underline{B}}_{p^t} \wedge \underline{\underline{N}}_2\right)\underline{\underline{A}}$ by $\underline{\underline{T}}$. Thus $\underline{\underline{V}} \leq \underline{\underline{AT}}$.

4.3 Proof of Theorem 4.1.1

4.3.A. $\underline{\underline{V}}$ *is generated by a finitely generated group.*

Proof. Let $A_0 = \underline{\underline{T}}(G)$, $F_k = F_k(\underline{\underline{V}})$ and $A_k = \underline{\underline{T}}(F_k)$ $(k = 1, 2, 3, \ldots)$. Then A_0 is an abelian verbal subgroup of G and, since G has no verbal torsion subgroups, A_0 is torsion-free. Thus each A_k is torsion-free since it can be embedded in A_0 .

If $\underline{\underline{V}}$ were not generated by a finitely generated group, then each F_k would generate a proper subvariety of $\underline{\underline{V}}$ and these would then satisfy *ii)* of 4.1.1 - say $F_k \in \underline{\underline{B}}_{n(k)}\underline{\underline{N}}_{c(k)}\underline{\underline{B}}_{n(k)}$ $(k = 1, 2, 3, \ldots)$. We claim that we may suppose $c(k)$ to be independent of k . For, putting $B_k = \underline{\underline{B}}_{n(k)}(F_k)$ and $N_k = \underline{\underline{N}}_{c(k)}(B_k)$, B_k/N_k is a finitely generated nilpotent group. Thus the torsion subgroup, which we denote by T_k/N_k , is finite. We now have a normal series of F_k ,

$$E \le T_k \le B_k \le F_k ,$$

in which T_k is of finite exponent, B_k/T_k is torsion-free nilpotent and F_k/B_k is of finite exponent. Since $\underline{\underline{A}}^2 \nleq \underline{\underline{V}}$, 2.3.6 shows that B_k/T_k is of bounded class and so, with a suitable adjustment to $n(k)$, we may suppose that
$c(1) = \ldots = c(k) = \ldots = c$, say.

Now N_k is torsion while A_k is torsion-free and so $N_k \cap A_k = E$. Hence

$$F_k \in \underline{\underline{N}}_c\underline{\underline{B}}_{n(k)} \vee \underline{\underline{T}} \le \underline{\underline{N}}_d\underline{\underline{B}}_{n(k)} \vee \underline{\underline{N}}_d\underline{\underline{A}} \le \underline{\underline{N}}_d\left[\underline{\underline{B}}_{n(k)}, \underline{\underline{A}}\right] ,$$

where $d = c$ if $c > 1$ and $d = 2$ if $c = 1$. From the description of the laws of such a product variety, given as (3.1) of Graham Higman [13], it follows that $\underline{\underline{N}}_d[\underline{\underline{B}}_{n(k)}, \underline{\underline{A}}]$ has as a basis for its laws the word

$$w_k(x_1, \ldots, x_{3d+3}) = [b_1, \ldots, b_{d+1}]$$

where $b_i = \left[x_{3i-2}, x_{3i-1}, x_{3i}^{n(k)}\right]$. Thus, if F_{3d+3} does not generate $\underline{\underline{V}}$,

$$w_{3d+3}(y_1, \ldots, y_{3d+3}) = e$$

is true in F_{3d+3} (we regard F_k as having free generators $y_1, \ldots, y_k$). But then $w_{3d+3}(y_1, \ldots, y_{3d+3}) = e$ is also true in G and so the law w_{3d+3} holds in $\underline{\underline{V}}$; that is $\underline{\underline{V}} \leq \underline{\underline{N}}_d[\underline{\underline{B}}_{n(k)}, \underline{\underline{A}}]$.

n(k) Hence $\underline{\underline{V}} \leq \underline{\underline{N}}_d\underline{\underline{N}}_2\underline{\underline{B}}$ and so, by 4.2.1, $\underline{\underline{V}}$ could not be a counterexample. This completes the proof of 4.3.A.

We shall now suppose that $\underline{\underline{V}}$ is generated by a finitely generated free group F, and put $A = \underline{\underline{T}}(F)$.

4.3.B. $[a, cx^m] = e$ *for all* $x \in F$, $a \in A$, *where* c *and* m *are natural numbers depending on* $\underline{\underline{V}}$ *only*.

Proof. Let $x \in F$ and put $H = gp(A, x)$. Then H is metabelian and so $H \in \underline{\underline{V}} \wedge \underline{\underline{A}}^2$. But, by 2.3.3 and 2.3.4, $\underline{\underline{V}} \wedge \underline{\underline{A}}^2$ satisfies *ii)* of 4.1.1 - say $\underline{\underline{V}} \wedge \underline{\underline{A}}^2 \leq \underline{\underline{B}}_m\underline{\underline{N}}_c\underline{\underline{B}}_m$. Thus, if $a \in A$,

$$[a^m, cx^m]^m = e \ .$$

It follows, using the fact that A is a torsion-free abelian normal subgroup of F, that $[a, cx^m] = e$.

4.3.C. *Suppose that* H/K *is an* F-*normal,* q-*elementary abelian factor of* A. *Then, for some natural number* $r(q)$, *depending only on* q *and* $\underline{\underline{V}}$, $\underline{\underline{B}}_{r(q)}\wedge$ *centralises* H/K.

(F)

Proof. Suppose, then, that $h \in H$, $x \in F$. Since $H \le A$, we may apply 4.3.B to show that $[h, cx^m] = e$ and, trivially, $[h, cx^m] \in K$. Let j be the least natural number such that $q^j \ge c$. Then, also trivially, $[h, q^j x^m] \in K$. Since H and K are normal in F and H/K is elementary abelian of exponent q,

$$\left[h, x^{q^j m}\right] \in K \ .$$

Hence, since x was arbitrary in F, putting $r(q) = q^j m$ gives the desired result. Since $r(q)$ depends on q, c and m only and c and m depend on $\underline{\underline{V}}$ only, $r(q)$ depends only on q and $\underline{\underline{V}}$.

The next three parts of this section will obtain information on A as an abelian group so that in 4.3.G we may extract a relevant property of $\mathrm{Aut}(A)$. We define D_n as $\bigcap_j \underline{\underline{B}}_{n^j}(A)$ where n is a natural number and the intersection is taken over all natural numbers j; we also define D as $\bigcap_n D_n$ where the intersection is taken over all natural numbers n. It is an elementary fact for all torsion-free abelian groups A that A/D_n and A/D are torsion-free.

4.3.D. *If* q *is a prime and* $q \neq p$ $\left(\textit{recall that } \underline{\underline{T}} = \left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{p^t}\right)\underline{\underline{A}}\right)$, *then* $D_q \neq E$.

Proof. Denote $\underline{\underline{B}}_{r(q)}(F)$ by H and $\underline{\underline{T}}(H)$ by B . Then H is verbal in F , B is verbal in H , and $\underline{\underline{B}}_{q^j}(B)$ is verbal in B $(j = 0, 1, 2, \ldots)$. (We shall, in the following, ease the notation by writing q^jB rather than $\underline{\underline{B}}_{q^j}(B)$ - and similarly for corresponding subgroups of A .) In particular, the elements of $\{q^jB/q^{j+1}B \mid j = 0, 1, 2, \ldots\}$ are F-verbal, and so F-normal, factors of A (since $B \leq A$) . Thus they are centralised by H .

Hence

$$H' \geq B \geq qB \geq \ldots \geq q^jB$$

gives a descending series of H'/q^jB with the first factor nilpotent and all other factors central, showing that H'/q^jB is nilpotent. Since H'/q^jB is also of finite exponent p^tq^j , it is the direct product of its Sylow subgroups - S_p, S_q , say. But $S_q = B/q^jB$ and $S_p \cong \left(H'/q^jB\right)/S_q = \left(H'/q^jB\right)/\left(B/q^jB\right) \cong H'/B$. Thus S_q is abelian and S_p has class at most 2 . Hence H'/q^jB has class at most 2 and so

$$[H', \underline{\underline{A}}^2(H)] \leq q^j B \quad (j = 0, 1, 2, \ldots) .$$

If $D_q = E$, then, since $B \leq A$, $\bigcap_j q^j B = E$. Thus $[H', \underline{\underline{A}}^2(H)] \leq \bigcap_j q^j B = E$ and $H \in \underline{\underline{N}}_2\underline{\underline{A}}$. Since H generates $\underline{\underline{V}}$, 4.2.1 shows that $\underline{\underline{V}}$ could not be a counterexample as we supposed. The proof of 4.3.D is complete.

4.3.E. $D_p = D$.

Proof. Trivially, $D \leq D_p$ and so it remains to prove that $D_p \leq D$. We shall show that A/D_q $(q \neq p)$ is free abelian of finite rank. Then we have

$$\bigcap_i p^i (A/D_q) = E$$

or $\bigcap_i p^i A \leq D_q$ $(i \in \underline{N})$. Hence $D_p \leq D_q$ for all primes q and so $D_p \leq \bigcap_q D_q = D$, where this intersection is taken over all primes q .

Now, by 4.3.D, D_q is a non-trivial verbal subgroup of F . Thus F/D_q generates a proper subvariety of $\underline{\underline{V}}$ which therefore satisfies *ii)* of 4.1.1; say, $F/D_q \in \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$. Let $M \leq F$ be such that $M/D_q = \underline{\underline{N}}_c\underline{\underline{B}}_n(F/D_q)$. Then $M/D_q \in \underline{\underline{B}}_n$ while A/D_q is torsion-free, and so $A \cap M = D_q$ and $A/D_q \cong AM/M$. But F/M is a finitely generated group in $\underline{\underline{N}}_c\underline{\underline{B}}_n$ and so is polycyclic. Therefore $AM/M \cong A/D_q$ is finitely generated. Thus A/D_q is free abelian of finite rank (as we have already commented that it is torsion-free).

4.3.F. $|A : pA|$ *is finite.*

Proof. We have shown, in 4.3.C, that, for some natural number $r(p)$, $\underline{\underline{B}}_{r(p)}(F)$ centralises A/pA. It is not difficult to extend this to the result that, for each natural number j, $\underline{\underline{B}}_{r(p^j)}(F)$, with $r(p^j) = p^{j-1}r(p)$ centralises A/p^jA (the method is similar to that used in the inductive step in the proof of 2.3.5 and we shall not include it here).

Denote $\left(A.\underline{\underline{B}}_{r(p^j)}(F)\right)/p^jA$ by H_j. Then A/p^jA is central in H_j. Also $\underline{\underline{T}}(H_j) \leq A/p^jA$ and so $H_j \in [\underline{\underline{E}}, \underline{\underline{T}}] \leq \underline{\underline{N}}_3\underline{\underline{A}} \wedge \underline{\underline{V}}$. Thus, by 4.2.1, $\text{var}\left(\{H_j \mid j \in \underline{N}\}\right)$ satisfies *ii)* of 4.1.1 - say $H_j \in \underline{\underline{B}}_n\underline{\underline{N}}_c\underline{\underline{B}}_n$ $(j \in \underline{N})$.

Suppose that $P_j \leq F$ is such that $P_j/p^jA = \underline{\underline{N}}_c\underline{\underline{B}}_n(H_j)$ and let $n = p^k.m$ where $p \nmid m$. We claim that $P_{k+1} \cap A \leq pA$. For, suppose $a \in P_{k+1} \cap A$. Then $a^n = \left(a^{p^k}\right)^m \in p^{k+1}A$ and so $a^{p^k} \in p^{k+1}A$. Hence, for some $b \in A$, $a^{p^k} = b^{p^{k+1}}$. Since A is torsion-free, $a = b^p$, and so $a \in pA$ as required.

Now AP_{k+1}/P_{k+1} is a subgroup of H_{k+1}/P_{k+1} which is a polycyclic group. Thus $A/\left(A \cap P_{k+1}\right) \cong AP_{k+1}/P_{k+1}$ is finitely generated. Since $pA \geq A \cap P_{k+1}$, A/pA is also finitely generated and so is finite.

4.3.G. The theorem will follow from the following result, which we state as an independent lemma, and which is due to L.G. Kovács.

LEMMA. *Suppose that* A *is a torsion-free abelian group, that* $\bigcap_j p^j A = E$ *for some prime* p, *and that* A/pA *is finite. Then every* p-*group of automorphisms of* A *is finite.*

Proof. First observe that if A/pA is finite, then so is A/p^2A. There is a natural homomorphism from the automorphism group of A to that of A/p^2A: it is clearly sufficient to show that the kernel of this homomorphism contains no automorphisms of order p. That is, that if $\beta^p = 1$ and $A(\beta-1) \le p^2A$ for some automorphism β of A, then $\beta = 1$ (1 being the identity automorphism). We show that if $A(\beta-1) \le p^kA$ for some $k > 1$ then also $A(\beta-1) \le p^{k+1}A$; it will then follow that $A(\beta-1) \le \bigcap_k p^kA = E$ and so $\beta = 1$. To this end, let $\beta^p = 1$, $k > 1$, and $A(\beta-1) \le p^kA$. Working in the endomorphism ring of A,

$$1 = \beta^p = \big(1+(\beta-1)\big)^p = 1 + p(\beta-1) + \sum_{i=2}^{p} \binom{p}{i}(\beta-1)^i ,$$

hence

$$p(\beta-1) = -\sum_{i=2}^{p} \binom{p}{i}(\beta-1)^i$$

and so $pA(\beta-1) \leq A(\beta-1)^2 \leq p^{2k}A \leq p^{k+2}A$. Since A is torsion-free, this implies that $A(\beta-1) \leq p^{k+1}A$, as claimed.

4.3.H. The proof of the theorem now follows easily. Firstly, we note that in view of 4.2.D, condition *i)* of 4.1.1 implies that $F_\infty(\underline{\underline{V}})$, and so A , is residually finite (17.81 of [19]). Thus $D = E$ and so, by 4.3.E, $D_p = E$. (We note that this is the first time we have used *i)* of 4.1.1.) Then A satisfies the conditions of the preceeding lemma.

Let C be the centraliser of A in F and put $H = F'/(F' \cap C)$. Then $H \cong F'C/C$ and $F'C/C$ is a p-group of automorphisms of A : so, by the preceding lemma, H is finite. Now H is the derived group of $F/(F' \cap C)$ and so, by 5.41 of B.H. Neumann [17], the centre of the latter group has finite index. Since $F' \cap C \in \underline{\underline{N}}_3$, we then have that $F \in \underline{\underline{N}}_3\underline{\underline{A}}\underline{\underline{B}}_m$, for some natural number m , and so, by 4.2.1, $\underline{\underline{V}}$ cannot be a counterexample.

With this final contradiction, Theorem 4.1.1 is proved.

CHAPTER 5

SOME VARIETIES WHICH DO NOT CONTAIN $\underline{\underline{C}}_p$

5.1 Statement of the Theorem

The purpose of this chapter is to prove

5.1.1 THEOREM. *Suppose that* $\underline{\underline{V}} \leq [\underline{\underline{E}}, \underline{\underline{A}}\underline{\underline{N}}_c]$ *for some natural number* c *and that* $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ *for all primes* p . *Then there exist natural numbers* d *and* m *such that* $\underline{\underline{V}} \leq \underline{\underline{A}}\underline{\underline{N}}_d\underline{\underline{B}}_m$.

The theorem will be proved in Section 5.3. In Section 5.2 we shall prove three lemmas, the first of which is designed largely to facilitate the proof of the theorem. The second, 5.2.2, will show that $\underline{\underline{C}}_p$ is minimal with respect to not being abelian by nilpotent by finite exponent (that it is not in fact abelian by nilpotent by finite exponent follows from results of P. Hall in, for example, [8]). The main purpose of this lemma, however, is to enable us to prove 5.2.4, which states that T_p cr $C(\infty)$ generates $\underline{\underline{C}}_p$ and which is of considerable interest in its similarity to a consequence of 2.3.7 - that $C(p)$ wr $C(\infty)$ generates $\underline{\underline{A}}_p\underline{\underline{A}}$.

far The question arises as to how ~~for~~ this similarity extends; that is, do crown products generate varieties of the form 'centre extended by ...' in a manner similar to that in which wreath products generate product varieties (see, for example, Section 2.2 of [19])? We would, of course, like to give proofs similar to those of

2.2 of [19] to produce such results. Unfortunately, our inability to form a useful parallel to unrestricted wreath products prevents this (see Section 4.3 of M.F. Newman [22] for a discussion of the similar problem for central products). In Lemma 5.2.4, we circumvent the problem in this special case by comparing some properties of T_p cr $C(\infty)$ and $\underline{Z}_p$.

$\lneqq_p$

5.1.2 REMARK. By Theorem 3.1 of Graham Higman [13], all varieties of the set $\{\underline{\underline{A}}\underline{\underline{N}}_c\underline{\underline{B}}_m \mid c, m \in \underline{N}\}$ are finitely based. Hence, by 2.3.10, every variety which is not itself abelian by nilpotent by finite exponent possesses subvarieties minimal with respect to not being abelian by nilpotent by finite exponent.

Finally, we introduce some notation which we shall use both in this chapter and the next, usually without further comment. Let G be a group of $\underline{\underline{N}}_2\underline{\underline{N}}_c$ for some natural number c and denote $\underline{\underline{N}}_c(G)$ by N .

As we have noted in Chapter 2, $N/Z(N)$ has a natural $\underline{Z}G$-module structure (in fact, even a $\underline{Z}(G/N)$-module structure; we shall use these two structures interchangeably where no confusion is likely to arise). Thus, if $u_1, u_2 \in N$ and $f \in \underline{Z}G$, $(u_2Z(N))^f$ is well-defined. If $u_3 \in (u_2Z(N))^f$ we shall define $[u_1, u_2^f]$ to be $[u_1, u_3]$; as N is a nilpotent group of class 2 , this definition does not depend on the choice of u_3 within $(u_2Z(N))^f$.

We have adopted this notation largely as a convenient 'shorthand'. There is, however, an alternative way of considering it which may clarify some of the manipulations in what follows. The map from $N/Z(N) \times N/Z(N)$ to $Z(N)$ defined by

$$(u_1 Z(N),\ u_2 Z(N)) \mapsto [u_1,\ u_2]$$

is an antisymmetric bilinear form on the $\underline{Z}G$-module $N/Z(N)$ which takes values in the abelian group $Z(N)$. The justification for our notation and the work we expect it to do now follows easily when we note that $\underline{Z}G$ is a set of linear functionals on $N/Z(N)$. Of course, if N has prime exponent p we may replace $\underline{Z}$ by $\underline{GF}(p)$ in the above.

If, further $G \in [\underline{\underline{E}},\ \underline{\underline{AN}}_c]$ and $g \in G$, then

$$\left[u_1,\ u_2^g\right] = \left[u_1^{g^{-1}},\ u_2\right].$$

Hence, if $f = \sum a_i g_i \in \underline{Z}G$, where $a_i \in \underline{Z}$, $g_i \in G$,

$$\left[u_1,\ u_2^f\right] = \left[u_1,\ \prod u_2^{a_i g_i}\right] = \prod\left[u_1,\ u_2^{g_i}\right]^{a_i} = \prod\left[u_1^{g_i^{-1}},\ u_2\right]^{a_i} =$$

$$\left[\prod u_1^{a_i g_i^{-1}},\ u_2\right] = \left[u_1^{f_1},\ u_2\right]$$

where $f_1 = \sum a_i g_i^{-1}$.

5.2 Three Lemmas

5.2.1 LEMMA. *Suppose that* $\underline{\underline{V}} \le [\underline{\underline{E}},\ \underline{\underline{AN}}_c]$ *for some natural number* c *and that all the finitely generated groups of* $\underline{\underline{V}}$ *have* max-n. *Then there exist natural numbers* d *and* m *such that*

$\underline{\underline{V}} \leq \underline{\underline{AN}}_d\underline{\underline{B}}_m$.

Proof. The proof will be by contradiction. Thus, by Remark 5.1.2, we may suppose that $\underline{\underline{V}}$ is a minimal counterexample to the lemma. Denote $F_\infty(\underline{\underline{V}})$ by F and let $y_1, \ldots, y_n, \ldots$ be a free generating set of F ; evidently $\underline{\underline{V}} \geq \underline{\underline{A}}$ and so the y_n have infinite order. The proof now splits into two cases.

a) Suppose that N is torsion-free.

Let G_1 be the subgroup of F generated by $y_1, \ldots, y_{2c+3}$; then G_1 is $\underline{\underline{V}}$-free on this generating set. Denote $\underline{\underline{AN}}_c(G_1)$ by A_1 . Then A_1 is central in G_1 and so every subgroup of A_1 is normal in G_1 . However, G_1 , as a finitely generated group of $\underline{\underline{V}}$, has max-n . Hence A_1 is a finitely generated abelian group. Denote $[y_1, \ldots, y_{c+1}]$ by w_1 , $[y_{c+2}, \ldots, y_{2c+2}]$ by w_2 and y_{2c+3} by y . Then the set $\left\{\left[w_1, w_2^{y^i}\right] \mid i \in \underline{Z}\right\}$ is an infinite subset of A_1 and so cannot be linearly independent over $\underline{Z}$. Thus, for some non-zero element $f(y)$ of $\underline{Z}(gp(y))$,

$$\left[w_1, w_2^{f(y)}\right] = e \ .$$

Now this relation holds identically in F , and, if we denote $Z(N)$ by Z , evidently implies that $w_2^{f(y)} \in Z$. But, by 2.3.1, Z is verbal in F and so $[x_1, \ldots, x_{c+1}]^{f(x_{c+2})}$ is a law of F/Z . This law has a consequence, obtained on replacing each of

$x_4, \ldots, x_{c+2}$ by x_3, of the form $[x_1, x_2]^{h(x_3)} d$ where $0 \neq (x_3 - 1)^{c-1} f(x_3) = h(x_3) \in \underline{Z}\big[gp(x_3)\big]$ and $d \in \underline{\underline{A}}^2(X)$. Hence, by 2.3.9, $\underline{\underline{A}}^2 \not\leq \mathrm{var}(F/Z)$ and so, by 4.2.1, $F/Z \in \underline{\underline{B}}_m \underline{\underline{N}}_d \underline{\underline{B}}_m$ for some natural numbers d and m.

Let M be the verbal subgroup of F defined by $M/Z = \underline{\underline{N}}_d \underline{\underline{B}}_m (F/Z)$. Then $M/Z \in \underline{\underline{B}}_m$. However, since N is torsion-free, so also is N/Z (see, for example, Lemma 1.63 of [25]). Hence $M \cap N = Z$. But $F/M \cap N \in \underline{\underline{N}}_d \underline{\underline{B}}_m \vee \underline{\underline{N}}_c \leq \underline{\underline{N}}_{\max(c,d)} \underline{\underline{B}}_m$. Hence, since $M \cap N$ is abelian, $\underline{\underline{V}} \leq \underline{\underline{A}}\underline{\underline{N}}_{\max(c,d)} \underline{\underline{B}}_m$ and $\underline{\underline{V}}$ cannot be a counterexample in this case.

b) Suppose that N has a non-trivial torsion subgroup.

Then the torsion subgroup of N is fully invariant in N and so verbal in $\underline{\underline{V}}$. Hence there is an elementary abelian subgroup P of N, of exponent p say, which is non-trivial and verbal in F. Thus F/P generates a proper subvariety of $\underline{\underline{V}}$ and so, since $\underline{\underline{V}}$ is a minimal counterexample to the lemma, $F/P \in \underline{\underline{A}}\underline{\underline{N}}_d \underline{\underline{B}}_m$ for some natural numbers d and m. Let B be the verbal subgroup of F defined by $B/P = \underline{\underline{B}}_m (F/P)$. Since $F/B \in \underline{\underline{B}}_m$ and $\underline{\underline{V}}$ is a minimal counterexample to the lemma, B generates $\underline{\underline{V}}$. Thus $\underline{\underline{V}} = \mathrm{var}\, B \leq \underline{\underline{A}}_p \underline{\underline{A}}\underline{\underline{N}}_d$. Let $e = \max(c, d)$. Then $\underline{\underline{V}} \leq \underline{\underline{A}}_p \underline{\underline{A}}\underline{\underline{N}}_e \wedge [\underline{\underline{E}}, \underline{\underline{A}}\underline{\underline{N}}_e]$.

3 Let G_2 be the subgroup of F generated by $y_1, \ldots, y_{2e+1}$; then G_2 is $\underline{\underline{V}}$-free on this generating set. Denote $\underline{\underline{A}}\underline{\underline{N}}_e(G_2)$ by

A_2 . Then A_2 is a central subgroup of G_2 , which has max-n . Hence A_2 is finitely generated. Since it is also an elementary abelian group, it is finite. Denote $[y_1, \ldots, y_{e+1}]$ by v_1 , $[y_{e+2}, \ldots, y_{2e+2}]$ by v_2 and y_{2e+3} by z . Then the set $\left\{\left[v_1, v_2^{z^i}\right] \mid i \in Z\right\}$ is an infinite subset of A and so cannot consist of distinct elements. In particular, for some $i \neq j$, $\left[v_1, v_2^{z^i-z^j}\right] = e$.

Now this relation holds identically in F , and, if we denote $\underline{\underline{N}}_e(F)$ by N_0 and $Z(N_0)$ by Z_0 , evidently implies that $v_2^{z^i-z^j} \in Z_0$. But, by 2.3.1, Z_0 is verbal in F and so $[x_1, \ldots, x_{e+1}]^{x_{e+2}^i - x_{e+2}^j}$ is a law of F/Z_0 . This law has a consequence $[x_1, x_2]^{h(x_3)}d$ where

$$0 \neq (x_3-1)^{e-1}\left(x_3^i-x_3^j\right) = h(x_3) \in \underline{Z}\left(gp(x_3)\right) ,$$

p is not a factor of $h(x_3)$ and $d \in \underline{A}^2(X)$. Then, by 2.3.9, $\underline{\underline{A}}_p\underline{\underline{A}} \nsubseteq \mathrm{var}(F/Z_0)$ and so, by *ii)* of 3.1.1, $F/Z_0 \in \underline{\underline{N}}_d\underline{\underline{B}}_m$ for some natural numbers d and m . Since Z_0 is abelian, $\underline{\underline{V}} \leq \underline{\underline{A}}\underline{\underline{N}}_d\underline{\underline{B}}_m$ and cannot be a counterexample to the lemma, which completes the proof.

5.2.2 LEMMA. *If* $\underline{\underline{V}}$ *is a proper subvariety of* $\underline{\underline{C}}_p$ *, then there exist natural numbers* c *and* m *such that* $\underline{\underline{V}} \leq \underline{\underline{A}}\underline{\underline{N}}_c\underline{\underline{B}}_m$.

5.2.3 COROLLARY. *Every finitely generated group in* $\underline{\underline{C}}_p$ *which does not generate* $\underline{\underline{C}}_p$ *, is residually finite.*

(NOTE: The result of 5.2.3, which is all we require for the proof of 5.2.4, may be deduced from Theorems 1 and 3 of C.K. Gupta, N.D. Gupta and A.H. Rhemtulla [5]. The basic method of proof of 5.2.2 is also similar to that of Theorem 2 of [5]. However, for the sake of completeness, since we believe our methods offer some improvement on those in that paper and since the result we obtain is considerably stronger in this case, we offer a separate proof.)

Proof. Firstly we note that the corollary follows immediately from the proof of the lemma and Theorem 1 of P. Hall [11].

By *ii)* of 3.1.1, we may suppose that $\underline{\underline{A}}_p\underline{\underline{A}} \leq \underline{\underline{V}}$. Let X_2 be the free word group of rank 2 on free generators x_1, x_2 and denote $[x_1, x_2]$ by z . By results of J.N. Ridley [24] (1.1 and the preceding paragraph),

$$X_2' = \mathrm{gp}\left(\left\{z^{x_1^i x_2^j} \mid i, j \in \underline{Z}\right\}\right)$$

and

$$\underline{\underline{A}}^2(X_2) = \mathrm{gp}\left(\left\{\left[z, z^{x_1^i x_2^j}\right] \mid i, j \in \underline{Z}\right\}\right)[\underline{\underline{E}}, \underline{\underline{A}}^2](X_2)$$

where $i \geq 0$ and $j > 0$ if $i = 0$.

Let p be an odd prime. Then

$$\underline{\underline{C}}_p = [\underline{\underline{E}}, \underline{\underline{A}}^2] \wedge \underline{\underline{B}}_p\underline{\underline{A}} \quad \text{and} \quad \underline{\underline{C}}_p \wedge \underline{\underline{A}}^2 = \underline{\underline{A}}_p\underline{\underline{A}} = \underline{\underline{A}}^2 \wedge \underline{\underline{B}}_p\underline{\underline{A}} .$$

Hence

$$\underline{\underline{C}}_p(X_2) = [\underline{\underline{E}}, \underline{\underline{A}}^2](X_2)\underline{\underline{B}}_p(X_2')$$

and

$$\underline{\underline{A}}_p\underline{\underline{A}}(X_2) = \underline{\underline{A}}^2(X_2)\underline{\underline{B}}_p(X_2') .$$

non-zero Thus, by the above, every non-zero element of $\underline{\underline{A}}_p\underline{\underline{A}}(X_2)/\underline{\underline{C}}_p(X_2)$ is of the form

$$[z, z^f]\underline{\underline{C}}_p(X_2)$$

with $f = \sum a_i x_1^{k_i} x_2^{\ell_i} \neq 0$, $a_i \in \underline{GF}(p)$, $k_i, \ell_i \in \underline{Z}$, $k_i \geq 0$, and $\ell_i > 0$ if $k_i = 0$. Now $\underline{\underline{A}}_p\underline{\underline{A}} \leq \underline{\underline{V}} < \underline{\underline{C}}_p$ and so $\underline{\underline{A}}_p\underline{\underline{A}}(X_2) \geq \underline{\underline{V}}(X_2) > \underline{\underline{C}}_p(X_2)$. Thus $\underline{\underline{V}}(X_2)$ contains an element

$$[z, z^{f_1}]c \quad \left(c \in \underline{\underline{C}}_p(X_2)\right)$$

where f_1 is a non-zero element of $\underline{GF}(p)(X_2)$ satisfying the conditions for f above. Thus $\underline{\underline{V}}$ has a law of this form.

Let $p = 2$. We recall that in a group of exponent 4 and class 2, the derived group is of exponent 2 and squares of elements are central. Now

$$\underline{\underline{C}}_2 = [\underline{\underline{E}}, \underline{\underline{A}}^2] \wedge \underline{\underline{B}}_4\underline{\underline{A}} \quad \text{and} \quad \underline{\underline{C}}_2 \wedge \underline{\underline{A}}^2 = \underline{\underline{A}}_4\underline{\underline{A}} = \underline{\underline{A}}^2 \wedge \underline{\underline{B}}_4\underline{\underline{A}} .$$

Hence

$$\underline{\underline{C}}_2(X_2) = [\underline{\underline{E}}, \underline{\underline{A}}^2](X_2)\underline{\underline{B}}_4(X_2')$$

and

$$\underline{\underline{A}}_4\underline{\underline{A}}(X_2) = \underline{\underline{A}}^2(X_2)\underline{\underline{B}}_4(X_2') .$$

Before we proceed with the proof, we note that, since squares of elements in $X_2'/\underline{\underline{C}}_2(X_2)$ are central, the group they generate has a natural $\underline{GF}(2)(X_2/X_2')$-module structure. Thus, when we write $(z^2)^g\underline{\underline{C}}_2(X_2)$, or $(z^2)^g\underline{\underline{A}}_4\underline{\underline{A}}(X_2)$, we refer to this structure; we do not mean $(z)^{2g}$.

Now $\underline{\underline{A}}_2\underline{\underline{A}}(X_2) = \underline{\underline{A}}^2(X_2)\underline{\underline{B}}_2(X_2')$ and so $\underline{\underline{A}}_2\underline{\underline{A}}(X_2)/\underline{\underline{A}}_4\underline{\underline{A}}(X_2)$ consists

(a) of elements of the form $(z^2)^g\underline{\underline{A}}_4\underline{\underline{A}}(X_2)$ where $g \in \underline{GF}(X_2/X_2')$. Also, from the above, $\underline{\underline{A}}_4\underline{\underline{A}}(X_2)/\underline{\underline{C}}_2(X_2)$ consists of elements of the form

$$[z, z^f]\underline{\underline{C}}_2(X_2)$$

where $f = \sum a_i x_1^{k_i} x_2^{\ell_i} \neq 0$, $a_i \in \underline{GF}(2)$, $k_i, \ell_i \in \underline{Z}$, $k_i \geq 0$,

and $\ell_i \gneq 0$ if $k_i = 0$.

Now $\underline{\underline{A}}_2\underline{\underline{A}} \leq \underline{\underline{V}} < \underline{\underline{C}}_2$ and so $\underline{\underline{A}}_2\underline{\underline{A}}(X_2) \geq \underline{\underline{V}}(X_2) > \underline{\underline{C}}_2(X_2)$. Hence $\underline{\underline{V}}(X_2)$ contains an element

$$[z, z^{f_1}](z^2)^{g_1}$$

where f_1 and g_1 are defined like f and g above and $f_1 \neq 0$. Thus $\underline{\underline{V}}$ has a law of this form.

Denote $F_\infty(\underline{\underline{V}})$ by F. In the element f_1 introduced above, whether p is odd or even, we may evidently suppose that, if $i \neq j$, $(k_i, \ell_i) \neq (k_j, \ell_j)$. Let ℓ be a natural number greater than

$$\max\left\{ |\ell_i| + |\ell_j| \;\middle|\; a_i, a_j \neq 0 \right\} .$$

Let u be an arbitrary element of F' and y an arbitrary free generator of F. Then we may substitute $y^\ell u$ for x_1 and y for x_2 in the laws obtained above to give the following relations in F:

$$\text{i)} \quad \left[[y^\ell u, y], [y^\ell u, y]^{f_2(y)} \right] = e \text{ , if } p \text{ is odd,}$$

$$\text{ii)} \quad \left[[y^\ell u, y], [y^\ell u, y]^{f_2(y)} \right] \left([y^\ell u, y]^2 \right)^{g_2(y)} = e \text{ , if } p \text{ is}$$

even; here, $f_2(y) = \sum a_i y^{k_i \ell + \ell_i}$ and, if $k_i \ell + \ell_i = k_j \ell + \ell_j$, then $|k_i - k_j| \ell = |\ell_i - \ell_j| \leq |\ell_i| + |\ell_j| < \ell$ and so $k_i = k_j$, $\ell_i = \ell_j$ implying that $i = j$. In particular, $f_2(y)$ is non-zero and all the powers of y involved in it have positive exponent.

We now expand relation i) above, and obtain

$$\left[[u, y], [u, y]^{f_2(y)} \right] = \left[u^{y-1}, u^{(y-1)f_2(y)} \right] = \left[u, u^{f_3(y)} \right] = e ,$$

where $f_3(y) = (y^{-1}-1)(y-1)f_2(y)$ and therefore $f_3(y) \notin \underline{GF}(p)$ and all the powers of y involved have non-negative exponent. By a similar method, relation ii) becomes

$$\left[u, u^{f_3(y)}\right](u^2)^{g_3(y)} = e$$

where $g_3(y) = (y-1)g_2(y)$.

Let u_1, u_2 be arbitrary elements of F' . Since u is an arbitrary element of F' , we may replace u by u_1u_2 in the relations above. When p is odd, we obtain

$$\begin{aligned}
e &= \left[u_1u_2, (u_1u_2)^{f_3(y)}\right] \\
&= \left[u_1, u_1^{f_3(y)}\right]\left[u_2, u_2^{f_3(y)}\right]\left[u_1, u_2^{f_3(y)}\right]\left[u_2, u_1^{f_3(y)}\right] \\
&= \left[u_1, u_2^{f_3(y)}\right]\left[u_2, u_1^{f_3(y)}\right] \\
&= \left[u_1, u_2^{f_3(y)}\right]\left[u_1, u_2^{-f_3(y^{-1})}\right] \\
&= \left[u_1, u_2^{f(y)}\right] ,
\end{aligned}$$

say, where $f(y) = f_3(y) - f_3(y^{-1}) \in \underline{GF}(p)(gp(y))$ and $f(y) \notin \underline{GF}(p)$.
In case $p = 2$, we obtain

$$e = \left[u_1u_2,\ (u_1u_2)^{f_3(y)}\right]\left((u_1u_2)^2\right)^{g_3(y)}$$

$$= \left(\left[u_1,\ u_1^{f_3(y)}\right]\left(u_1^2\right)^{g_3(y)}\right)\left(\left[u_2,\ u_2^{f_3(y)}\right]\left(u_2^2\right)^{g_3(y)}\right) \times$$

$$\left[u_1,\ u_2^{f_3(y)}\right]\left[u_2,\ u_1^{f_3(y)}\right][u_2,\ u_1]^{g_3(y)}$$

$$= \left[u_1,\ u_2^{f(y)}\right][u_2,\ u_1]^{g_3(y)} .$$

But, since $[u_2,\ u_1]$ is central in F and $(y-1)|g_3(y)$, $[u_2,\ u_1]^{g_3(y)} = e$. Hence, whether p is odd or even, F has a relation

$$\left[u_1,\ u_2^{f(y)}\right] = e$$

where u_1, u_2 are arbitrary elements of F', y is an arbitrary free generator of F and $f(y)$ is a non-scalar element of $\underline{GF}(p)\left(gp(y)\right)$.

Denote $Z(F')$ by Z. Then the relation obtained above shows that, for all elements d of F' and y of F, $d^{f(y)} \in Z$. Since, by 2.3.1, Z is verbal in F, $[x_1,\ x_2]^{f(x_3)}$ is a law in F/Z, where $f(x_3)$ is a non-zero element of $\underline{GF}(p)\left[gp(x_3)\right]$ (which we may regard as an element of $\underline{Z}\left[gp(x_3)\right]$ not divisible by p). Thus, by 2.3.9, $\underline{\underline{A}}_p\underline{\underline{A}} \nleq var(F/Z)$ and so, by *ii)* of 3.1.1, F/Z is nilpotent by finite exponent. Since Z is abelian, $\underline{\underline{V}}$ is abelian by nilpotent by finite exponent, as we claimed.

5.2.4 LEMMA. *The crown product* T_p cr $C(\infty)$ *generates* $\underline{\underline{C}}_p$.

Proof. Firstly, we must show that T_p cr $C(\infty) \in \underline{\underline{C}}_p$. Denote $F_2(\underline{\underline{C}}_p)$ $\left(\text{that is, } F_2\left(\underline{\underline{T}}_p\underline{\underline{A}} \wedge [\underline{\underline{E}}, \underline{\underline{A}}^2]\right)\right)$ by F and let x, y be a free generating set of F . If we denote $[x, y]$ by v , then, by the results of Ridley quoted on p. 65,

$$\Omega = \left\{\left[v, v^{x^i y^j}\right] \mid i, j \in \underline{Z}, i \geq 0 \text{ , and } j > 0 \text{ if } i = 0\right\} \cup$$
$$\cup \left\{\left(v^{x^i y^j}\right)^p \mid i, j \in \underline{Z}\right\} \setminus \{e\}$$

is a basis for $\underline{\underline{A}}_p(F')$. Denote $gp\left(v, v^y\right)$ by A and $gp\left(\Omega \setminus \left\{[v, v^y]\right\}\right)$ by B .

We claim that $gp(AB/B, xB) \cong T_p$ cr $C(\infty)$. By our choice of B , $[v, v^y] \notin B$ and so AB/B is a non-abelian group of $\underline{\underline{T}}_p$ generated by two elements of order p . Thus $AB/B \cong T_p$ and $Z(AB/B) = \underline{\underline{A}}_p(F')/B$. It will evidently suffice to show that the group generated by the conjugates of AB/B under powers of $\cancel{y}B$ is their central product amalgamating the centre. To show this, it will be enough to show that $\left[A^{x^i}B/B, A^{x^j}B/B\right] = E$ if $i \neq j$ and that the groups $\left(A^{x^i}B/B\right) / Z\left(A^{x^i}B/B\right)$ generate their direct product. The former is immediate from our choice of B and the latter follows from the fact that $Z\left(A^{x^i}B/B\right) = \underline{\underline{A}}_p(F')/B$ and that (again by Ridley, loc. cit.) the elements $v^{x^i y}\underline{\underline{A}}_p(F')$, $v^{x^i}\underline{\underline{A}}_p(F')$ form part of a free basis of $F'/\underline{\underline{A}}_p(F')$.

~~5.2.4 LEMMA. *The crown product* T_p cr $C(\infty)$ *generates* $\underline{\underline{C}}_p$.~~

~~Proof~~. By Corollary 5.2.3, it will suffice to show that T_p cr $C(\infty)$ is not residually finite. Denote it by T. We defined T by a homomorphism $T_p \text{ wr } C(\infty) \to T$ and we shall call the image of the base group under this homomorphism, the base group of T and denote it by B. Then B is a central product of copies of T_p amalgamating the centre. Thus, if A is the amalgamated subgroup of B, A is the centre of B, by Lemma 4.3.3 of [22]. Since $T/A \cong C(p) \text{ wr } C(\infty)$ which has trivial centre, by 24.23 of [19], A is also the centre of T. We also note that $T' \leq B$, $B' = A$ and the well known, and trivial, result that, if G is a group, then $H \trianglelefteq G$ and $H \cap G' = E$ implies that $H \leq Z(G)$.

We claim that A is a monolith of T. For, let $N \trianglelefteq T$ and and $A \nleq N$. Then $N \cap A = E$, since A is a minimal subgroup of T. Hence $(N \cap B) \cap B' = N \cap A = E$ and so, since $N \cap B \trianglelefteq B$, $N \cap B \leq Z(B) = A$. Thus, since $A \nleq N$, $N \cap B = E$ and so $N \cap T' = E$. Hence $N \leq Z(T) = A$ and so $N = E$. Thus T is a monolithic group and since it is not finite, it is not residually finite, as required.

5.3 Proof of Theorem 5.1.1

We shall prove the theorem by contradiction. By Remark 5.1.2, we may suppose that $\underline{\underline{V}}$ is a minimal counterexample to the theorem. If $\underline{\underline{V}}$ were not generated by a finitely generated group, then every finitely generated group of $\underline{\underline{V}}$ would generate a proper subvariety of $\underline{\underline{V}}$ and so, by 2.2.6, would have max-n (evidently, by 2.2.5, a finite extension of a group with max-n also has max-n). By 5.2.1, this would contradict the choice of $\underline{\underline{V}}$ as a counterexample to the theorem. Let $F = F_k(\underline{\underline{V}})$ $(k \geq c+2)$ be a group generating $\underline{\underline{V}}$ and let $y_1, \ldots, y_k$ be a free generating set of F .

Denote $\underline{\underline{N}}_c(F)$ by N and the torsion subgroup of N by T . Then T is fully invariant in N and so verbal in F . If T is non-trivial, it has a non-trivial subgroup of finite exponent, P say, which is fully invariant in T and so verbal in F . Hence, in this case, F/P generates a proper subvariety of $\underline{\underline{V}}$ and so has max-n . Thus T/P , the torsion subgroup of N/P , has finite exponent and so T has finite exponent. Thus, by 2.2.4, there is a natural number m such that $\underline{\underline{B}}_m(N) \cap T = E$. Thus $\underline{\underline{V}} \leq \mathrm{var}\big(F/\underline{\underline{B}}_m(N)\big) \vee \mathrm{var}(F/T)$. However, $\underline{\underline{V}}$, as a minimal counterexample to the theorem, evidently cannot be the join of two proper subvarieties. Hence, if T is non-trivial, $\underline{\underline{B}}_m(N) = E$.

It will only be necessary, therefore, to consider the two cases

a) N is torsion-free,

b) N is of finite exponent.

As N is nilpotent, it is, in case b), the direct product of its Sylow subgroups, each of which is fully invariant in N and so verbal in F . But $\underline{\underline{V}}$ evidently cannot be the join of any finite number of proper subvarieties. Hence all but one of these Sylow subgroups are trivial and so N has exponent a prime power, p^i say. Now, as $\underline{\underline{V}}$ is a minimal counterexample to the theorem, N'

is non-trivial and so has a non-trivial elementary abelian subgroup P which we may choose to be fully invariant in N' and so verbal in F. Thus F/P generates a proper subvariety of $\underline{\underline{V}}$ and so $F/P \in \underline{\underline{AN}}_d \underline{\underline{B}}_m$ for some natural numbers d and m. Let B be the verbal subgroup of F defined by $B/P = \underline{\underline{B}}_m(F/P)$. Then, since $F/B \in \underline{\underline{B}}_m$, and $\underline{\underline{V}}$ is a minimal counterexample, B generates $\underline{\underline{V}}$. Hence

$$\underline{\underline{V}} \leq \underline{\underline{B}}_{p^i}\underline{\underline{N}}_c \wedge \underline{\underline{A}}_p\underline{\underline{AN}}_d \wedge [\underline{\underline{E}}, \underline{\underline{AN}}_c] \leq \underline{\underline{B}}_{p^i}\underline{\underline{N}}_e \wedge \underline{\underline{A}}_p\underline{\underline{AN}}_e \wedge [\underline{\underline{E}}, \underline{\underline{AN}}_e]$$

where $e = \max(c, d)$. Therefore, if we can establish that no counterexample to the theorem can be contained in $\underline{\underline{B}}_{p^i}\underline{\underline{N}}_e \wedge \underline{\underline{A}}_p\underline{\underline{AN}}_e \wedge [\underline{\underline{E}}, \underline{\underline{AN}}_e]$ for an arbitrary natural number e, $\underline{\underline{V}}$ cannot be a counterexample. Hence, in place of case b), it will be sufficient to consider the case $\underline{\underline{V}} \leq \underline{\underline{B}}_{p^i}\underline{\underline{N}}_c \wedge \underline{\underline{A}}_p\underline{\underline{AN}}_c \wedge [\underline{\underline{E}}, \underline{\underline{AN}}_c]$ which, in the notation we have established, amounts to

b') N is of finite prime power exponent p^i and N' is elementary abelian.

The next step of the theorem will consist of showing that, to prove the theorem in case b'), it will be sufficient to consider the case

c) $N \in \underline{\underline{T}}_p$.

We shall suppose that the theorem is true in case c) and then show that it is true in case b'). Suppose, then, that $\underline{\underline{V}} \leq \underline{\underline{B}}_{p^i}\underline{\underline{N}}_c \wedge \underline{\underline{A}}_p\underline{\underline{AN}}_c \wedge [\underline{\underline{E}}, \underline{\underline{AN}}_c]$ and that $\underline{\underline{V}} \nleq \underline{\underline{B}}_{p^{i-1}}\underline{\underline{N}}_c$. By the hypothesis we have adopted, we may suppose that $i \geq 2$ if p is odd and $i \geq 3$ if $p = 2$. Now $\underline{\underline{V}}$ is a minimal counterexample and $\underline{\underline{V}} \wedge \underline{\underline{B}}_{p^{i-1}}\underline{\underline{N}}_c$ is a proper subvariety of V. Hence, by 2.2.6, the finitely generated groups of $\underline{\underline{V}} \wedge \underline{\underline{B}}_{p^{i-1}}\underline{\underline{N}}_c$ have max-n. To show that the theorem is true for $\underline{\underline{V}}$, it will be sufficient, by 5.2.1, to show that all finitely generated groups of $\underline{\underline{V}}$ have max-n.

Suppose, then, that G is a finitely generated group of $\underline{\underline{V}}$ and put $M = \underline{\underline{N}}_c(G)$, $L = \underline{\underline{B}}_{p^{i-1}}(M)$. Let $a, b \in M$. Now M is nilpotent of class at most 2. Hence

$$(ab)^{p^{i-1}} = a^{p^{i-1}} b^{p^{i-1}} [b, a]^{p^{i-1}(p^{i-1}-1)/2} = a^{p^{i-1}} b^{p^{i-1}} c,$$

say, where $c \in \underline{\underline{B}}_{p^{i-1}}(M')$ if p is odd and $c \in \underline{\underline{B}}_{2^{i-2}}(M')$ if $p = 2$. Now M' has prime exponent. Also, if p is odd, $i \geq 2$ and so $\underline{\underline{B}}_{p^{i-1}}(M') = E$, and if $p = 2$, $i \geq 3$ and so $\underline{\underline{B}}_{2^{i-2}}(M') = E$. Thus, in either case, $(ab)^{p^{i-1}} = a^{p^{i-1}} b^{p^{i-1}}$.

The above relation shows that every element of L is a

M p^{i-1}-th power of an element of $\cancel{N}$ and, since

M $\left[a^{p^{i-1}}, b\right] = [a, b]^{p^{i-1}} = e$, that L is central in $\cancel{N}$. Now G/L,

being a finitely generated group of $\underline{\underline{V}} \wedge \underline{\underline{B}}_{p^{i-1}}\underline{\underline{N}}_c$, has max-n.

M Hence $\cancel{N}/L$ is finitely generated as a normal subgroup of G/L -

M by $a_1L, \ldots, a_kL$, say. Thus, if $n \in \cancel{N}$, there are elements g_t

of G such that

$$n = \prod_t a_{i_t}^{\varepsilon_t g_t} \ell \quad (1 \le i_t \le k,\ \varepsilon_t = \pm 1)$$

where $\ell \in L$. But then

$$n^{p^{i-1}} = \prod_t a_{i_t}^{p^{i-1}\varepsilon_t g_t} \quad \left(\text{since } \ell^{p^{i-1}} = e\right) .$$

Thus L is finitely generated as a normal subgroup of G by the elements $a_1^{p^{i-1}}, \ldots, a_k^{p^{i-1}}$.

We have said that L is central in M . Hence, if we denote $C_G(L)$ by C , $C \geq M$ and so G/C is nilpotent. As L is finitely generated as a normal subgroup of G , it is also a finitely generated $\underline{Z}(G/C)$-module. However, by Theorem 1 of P. Hall [8], $\underline{Z}(G/C)$ has the maximal condition on its right ideals. Thus, by a well known result of module theory (see, for example, 11.16 of [4]), L has maximal condition on its $\underline{Z}(G/C)$-submodules, which are, of course, precisely the normal subgroups of G contained in L . That is, L has max-G and so, since G/L has max-n , 2.2.5 completes this part of the theorem.

Thus we need only prove the theorem in cases a) and c). Firstly we establish some notation common to both cases. Denote $[y_1, \ldots, y_{c+1}]$ by w and y_{c+2} by y . Denote $gp(w, w^y)$ by

3/3/3 A_0 and $A_0^{y^{2i}} = gp\left(w^{y^{2i}}, w^{y^{2i+1}}\right)$ by A_i . Denote $gp\left(\left\{\left[w, w^{y^i}\right] \mid i \in \underline{Z}\right\}\right)$ by D and $gp\left(\left\{\left[w, w^{y^i}\right] \mid i \in \underline{Z},\ |i| \neq 1\right\}\right)$ by B . Then B and D are central, and so normal, in F and

$[A_i, A_j] \leq B$ if $i \neq j$.

Suppose, firstly, that N has exponent an odd prime p . We shall show that there is a relation, in F , of the form

$$\left[w, w^{f_1(y)}\right] = e$$

where $o \neq f_1(y) = a_1 y + \ldots + a_n y^n \in \underline{GF}(p)\left(gp(y)\right)$. Suppose, then, that no such relation holds. In particular, $[w, w^y] \notin B$. Thus $A_i' \nleq B$ and so A_iB/B is non-abelian. Now $B \leq D \leq A_iB$ and D/B has order p . Also $A_oB/D = A_oD/D$ is a 2-generator abelian group of exponent p and so, of course, has order p^2 . Hence A_oB/B is a non-abelian group of order p^3 and exponent p . Thus $A_oB/B \cong T_p$ and $Z\left(A_oB/B\right) = D/B$.

3 We wish to show that $gp\left(A_oB/B, y^2B\right) \cong T_p$ cr $C(\infty)$. It will evidently suffice to show that the group generated by the conjugates

3 of A_oB/B under powers of y^2 is their central product amalgamating

3 their centre, D/B , and that y^2B centralises D/B . The latter is immediate, as D is central in F , and to show the former we need only show that $[A_iB/B, A_jB/B] = E$ if $i \neq j$ and

$A_iB/B \cap A^iB/B = D/B$, where $A^iB/B = gp\left(\{A_jB/B \mid j \in \underline{Z}, j \neq i\}\right)$. The first of these follows immediately from our comment that $[A_i, A_j] \leq B$ if $i \neq j$ and, if $a \in A^iB/B$ then a centralises A_iB/B and if, further, $a \in A_iB/B$, then $a \in Z\left(A_iB/B\right) = D/B$, 3 which established the second. Hence $gp\{A_oB/B, y^{\ell}B\} \cong T_p \text{ cr } C(\infty)$ and so, by 5.2.4, $\underline{\underline{C}}_p \leq \underline{\underline{V}}$, contradicting the assumptions of the theorem. Hence a relation of the type $\left[w, w^{f_1(y)}\right] = e$ holds in F.

Let u be an arbitrary element of N and let τ be the endomorphism of F defined by

$$\tau : y_1 \mapsto u , \quad \tau : y_i \mapsto y \quad (i \geq 2) .$$

Then

$$e = e\tau = \left[w, w^{f_1(y)}\right]\tau = \left[u^{(y-1)^c}, u^{(y-1)^c f_1(y)}\right] = \left[u, u^{f_2(y)}\right],$$

where

$$o \neq (y^{-1}-1)^c (y-1)^c f_1(y) = f_2(y) \in \underline{GF}(p)\{gp(y)\} .$$

Let u_1, u_2 be arbitrary elements of N. Then

$$\begin{aligned} e &= \left[u_1u_2, (u_1u_2)^{f_2(y)}\right] \\ &= \left[u_1, u_1^{f_2(y)}\right]\left[u_2, u_2^{f_2(y)}\right]\left[u_1, u_2^{f_2(y)}\right]\left[u_2, u_1^{f_2(y)}\right] \\ &= \left[u_1, u_2^{f_3(y)}\right] \end{aligned}$$

where

$$o \neq f_3(y) = f_2(y) - f_2(y^{-1}) =$$

$$(y^{-1}-1)^c(y-1)^c\left(f_1(y)-f_1(y^{-1})\right) \in \underline{GF}(p)(gp(y)) .$$

(This technique is evidently similar to one used in 5.2.2. We shall also have occasion to use it in the proofs of the remaining cases and we shall not there fill in the details.)

Therefore, for all elements u_1 and u_2 of N, $\left[u_1, u_2^{f_3(y)}\right] = e$. Denote $Z(N)$ by Z. Then this relation implies that, for all elements u of N, $u^{f_3(y)}Z = Z$ and so, that F/Z has a law $[x_1, \ldots, x_{c+1}]^{f_3(x_{c+2})}$. It follows that $[x_1, x_2]^{f_4(x_3)}d$ is a law in F/Z, where $f_4(x_3)$ is a non-zero element of $\underline{GF}(p)(gp(x_3))$ (which we may regard as an element of $\underline{Z}(gp(x_3))$ not divisible by p) and $d \in \underline{\underline{A}}^2(X)$. Hence, by 2.3.9, $\underline{\underline{A}}_p\underline{\underline{A}} \nsubseteq \text{var}(F/Z)$ and so, by *ii)* of 3.1.1, F/Z is nilpotent by finite exponent. Since Z is abelian, this contradicts the choice of $\underline{\underline{V}}$ as a counterexample and so completes the proof of case c) when p is odd.

Suppose now that N has exponent 4. Denote $gp\left(B, (w^2)^{y^i} \mid i \in \underline{Z}\right)$ by K and $gp\left(D, (w^2)^{y^i} \mid i \in Z\right)$ by L and note that both K and L are normalised by y and since, in a group of $\underline{\underline{T}}_2$, all squares are central, both K and L are central in N. We shall show that there is a relation in F of the form

$$\left[w, w^{f_1(y)}\right]\left(w^2\right)^{g(y)} = e$$

where $o \neq f_1(y) = a_1y + \ldots + a_ny^n \in \underline{GF}(2)\{gp(y)\}$ and $g(y) \in \underline{GF}(2)\{gp(y)\}$. It will then follow, using methods similar to those used in the case of odd p , and in the proof of 5.2.2, that $\underline{\underline{V}}$ cannot be a counterexample.

We shall suppose, then, that no relation of this type holds in F . In particular, therefore, $[w, w^y] \notin K$. Thus $A_o' \nleq K$ and so A_oK/K is non-abelian; since it is generated by two elements of order 2 and is in $\underline{\underline{T}}_2$, it is a dihedral group of order 8 . ~~Our definition of T_2 allows the interpretation that $T_2 \cong A_oK/K$ (we could have chosen T_2 to be the dihedral group of order 8 , but in doing so we would have lost some attractive generality; for example, in the statement of 5.2.4)~~. We may now proceed, as with the case of odd

3 p , to show that $gp\left(A_oK/K, y^{\cancel{f}\,2}K\right) \cong T_2 \text{ cr } C(\infty)$ and so show that a relation of the type described above does, in fact, hold in F . This completes the proof of case c).

Suppose, finally, that N is torsion-free. We shall show, as before, that there is a relation in F of the form

$$\left[w, w^{f_1(y)}\right] = e$$

where $o \neq f_1(y) = a_1y + \ldots + a_ny^n \in \underline{Z}(gp(y))$. Suppose that this is not the case. Then, in particular, D/B is torsion-free and if $(w^nw^{my}c)B$ $(c \in D)$ is an element of $Z(A_oB/B)$, then

$[w^n w^{my} c, w] = [w^{my}, w] \in B$ and so $m = 0$; similarly, $n = 0$ and so D/B is the centre of A_0B/B . By, for example, 1.63 of Derek J.S. Robinson [25], A_0B/B is torsion-free. Thus A_0B/B is a 2-generator, non-abelian torsion-free group of $\underline{\underline{N}}_2$ with centre D/B . Proceeding as with case c), we may now show that

3 $\mathrm{gp}\left(A_0B/B,\ y^{\ell}B\right) \cong (A_0B/B)\ \mathrm{cr}\ C(\infty)$.

Now, since A_0B/B is finitely generated and torsion-free, it is residually of prime exponent, by Theorem 1 of Graham Higman [12] and so, obviously residually a 2-generator monolithic (that is, cyclic centre) group of $\underline{\underline{T}}_p$, for some prime p . A group of the latter type must be either abelian or isomorphic to T_p and so, since A_0B/B is not abelian, there must be at least one prime p such that there is an epimorphism $A_0B/B \to T_p$. Thus, by 22.11 of [19], there is an epimorphism $A_0B/B\ \mathrm{cr}\ C(\infty) \to T_p\ \mathrm{cr}\ C(\infty)$ (the second half of the statement of 22.11 evidently guarantees that this result is applicable to crown products). Thus $T_p\ \mathrm{cr}\ C(\infty) \in \underline{\underline{V}}$ and so $\underline{\underline{C}}_p \leq \underline{\underline{V}}$, contradicting the assumptions of the theorem.

Thus a relation

$$\left[w, w^{f_1(y)}\right] = e$$

does, in fact, hold in F . Then, as in the proof of case c), we may show that, if u_1, u_2 are arbitrary elements of N ,

$\left[u_1, u_2^{f_3(y)}\right] = e$, where $f_3(y)$ is a non-zero element of

$\underline{Z}(gp(y))$. Denote $Z(N)$ by Z . Then $u_2^{f_3(y)} Z = Z$ and so

$[x_1, x_2]^{f_4(x_3)} d$ is a law in F/Z , where $o \neq f_4(x_3) \in \underline{Z}\big(gp(x_3)\big)$

and $d \in \underline{\underline{A}}^2(X)$. Thus, by 2.3.9, $\underline{\underline{A}}^2 \nsubseteq \mathrm{var}(F/Z)$ and so, by 4.2.1, $F/Z \in \underline{\underline{B}}_m \underline{\underline{N}}_d \underline{\underline{B}}_m$ for some natural numbers d and m . Let B be a

(F/Z) verbal subgroup of F defined by $B/Z = \underline{\underline{N}}_d \underline{\underline{B}}_m(\cancel{F})$. Then $B/Z \in \underline{\underline{B}}_m$.

But, since N is torsion-free, so also is N/Z (see, for example,

B 1.63 of [25]). Thus $\cancel{M} \cap N = Z$. But

B $$F/\cancel{M} \cap N \in \underline{\underline{N}}_d \underline{\underline{B}}_m \vee \underline{\underline{N}}_c \leq \underline{\underline{N}}_{\max(c,d)} \underline{\underline{B}}_m .$$

B Hence, since $\cancel{M} \cap N$ is abelian, $\underline{\underline{V}}$ cannot be a counterexample to the theorem. Thus the proof of case a), and, with it, the proof of the theorem, is complete. (The result as we stated it in the introduction, that is for $\underline{\underline{V}} \leq [\underline{\underline{E}}, \underline{\underline{AN}}_c]$ replaced by $\underline{\underline{V}} \leq [\underline{\underline{E}}, \underline{\underline{AN}}_c] \underline{\underline{B}}_n$ $(n \in \underline{N})$, evidently follows immediately.)

CHAPTER 6

A CRITERION FOR MAX-N

6.1 Introduction

The purpose of this chapter is to establish a partial dichotomy for the properties 'locally max-n' and 'locally residually finite'. We shall show that a metanilpotent variety $\underline{\underline{V}}$ will have these properties precisely if $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ for all primes p. The core of the proof will consist of showing the result for 'locally max-n' if $\underline{\underline{V}} \leq \underline{\underline{N}}_2\underline{\underline{A}}$. This will be done in Section 2. The deduction of the main theorems is then given in Section 3.

6.2 Subvarieties of $\underline{\underline{N}}_2\underline{\underline{A}}$

We shall prove the following:

6.2.1 PROPOSITION. *Let* $\underline{\underline{V}} \leq \underline{\underline{N}}_2\underline{\underline{A}}$ *and suppose that* $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ *for all primes* p. *Then all finitely generated groups in* $\underline{\underline{V}}$ *have* max-n.

Proof. If $\underline{\underline{V}}$ has finite exponent, then the finitely generated groups of $\underline{\underline{V}}$ are finite and so certainly have max-n. Thus we may suppose that $\underline{\underline{V}}$ has exponent zero. Evidently it will be sufficient to show that, if G is a finitely generated free group of $\underline{\underline{V}}$, then G has max-n.

Now, since $\underline{\underline{V}}$ has exponent zero, the elements of G of finite order lie in G' and so form a fully invariant subgroup T of G. By Theorem 5 of [16], G has the maximal condition on its fully invariant subgroups, and so T has finite exponent. Thus, by 2.2.4, there is a natural number m such that $\underline{\underline{B}}_m(G') \cap T = E$. Therefore $G \in \mathrm{var}\big(G/\underline{\underline{B}}_m(G')\big) \vee \mathrm{var}(G/T)$ and so, by 2.2.5, G will have max-n if the finitely generated groups of these varieties have max-n. Hence a proof of the proposition will follow from a proof in the cases

a) $\underline{\underline{V}}$ is torsion-free,

b) $\underline{\underline{V}} \leq \underline{\underline{B}}_m\underline{\underline{A}}$.

Since G' is nilpotent, in case b) it is the direct product of its Sylow subgroups. Hence case b) will evidently follow from

b') $\underline{\underline{V}} \leq \underline{\underline{B}}_{p^i}\underline{\underline{A}}$, where p is prime.

The next step of the proof will be to show that b') follows from

c) $\underline{\underline{V}} \leq \underline{\underline{T}}_p\underline{\underline{A}}$, where p is prime.

Suppose, then, that we have shown the theorem in case c); that is, we have shown that if H is a finitely generated group in $\underline{\underline{V}}$ and H' has exponent an odd prime p, or exponent 4, then H has max-n. This yields the starting point for an inductive proof and so we shall further suppose that $\underline{\underline{V}} \leq \underline{\underline{B}}_{p^i}\underline{\underline{A}}$ and that all finitely generated groups in $\underline{\underline{V}} \wedge \underline{\underline{B}}_{p^{i-1}}\underline{\underline{A}}$ have max-n where $i \geq 2$ if p is odd and $i \geq 3$ if $p = 2$.

Let G be a finitely generated group in $\underline{\underline{V}}$ and denote $\underline{\underline{B}}_{p^{i-1}}(G')$ by L. Suppose, for the time being, that $\underline{\underline{B}}_{p^j}\left(\underline{\underline{A}}^2(G)\right) = E$, where $j = i-1$ if p is odd and $j = i-2$ if $p = 2$. Now G' is nilpotent of class 2 and so, if $a, b \in G'$,

$$(ab)^{p^{i-1}} = a^{p^{i-1}} b^{p^{i-1}} [b, a]^k \quad \text{where} \quad k = p^{i-1}\left(p^{i-1}-1\right)/2 .$$

Hence, whether p is odd or even, $(ab)^{p^{i-1}} = a^{p^{i-1}} b^{p^{i-1}}$. Thus each element of L is a p^{i-1}-th power of an element of G' and so, since $\left[a^{p^{i-1}}, b\right] = [a, b]^{p^{i-1}} = e$, L is central in G'.

By our inductive hypothesis, G/L has max-n. Thus G'/L is finitely generated as a normal subgroup of G/L - by $a_1L, \ldots, a_kL$ say. Hence, if $b \in G'$, there are elements $g_1, \ldots, g_t, \ldots$ of G such that

$$b = \prod_t a_{i_t}^{\varepsilon_t g_t} \ell \quad \left(1 \le i_t \le k,\ \varepsilon_t = \pm 1\right)$$

where $\ell \in L$. But then

$$b^{p^{i-1}} = \prod_t a_{i_t}^{\varepsilon_t p^{i-1} g_t}, \text{ since } \ell^{p^{i-1}} = e ,$$

and so L is finitely generated, by $a_1^{p^{i-1}}, \ldots, a_k^{p^{i-1}}$, as a normal subgroup of G. Thus L is a finitely generated $\underline{Z}\left(G/C_G(L)\right)$-module. However, as L is central in G', $G/C_G(L)$ is a finitely generated abelian group and so, by Theorem 1 of [8],

$\underline{\underline{Z}}\{G/C_G(L)\}$ has maximal condition on its ideals. Hence, L has maximal condition on its $G/C_G(L)$-submodules. That is, L has max-G. Thus, by 2.2.5, G has max-n as required.

Suppose now that G is an arbitrary finitely generated group in $\underline{\underline{V}}$. Denote $\underline{\underline{B}}_{p^j}\{\underline{\underline{A}}^2(G)\}$ by K (j is the integer we defined above). Then we have shown that G/K has max-n and so $\underline{\underline{A}}^2(G)/K$ is finitely generated as a normal subgroup of G, by $a_1K, \ldots, a_nK$ say. Thus, if $b \in \underline{\underline{A}}^2(G)$, there are elements $g_1, \ldots, g_t, \ldots$, such that

$$b = \prod_t a_{i_t}^{\varepsilon_t g_t} \cdot k \quad \{1 \le i_t \le n,\ \varepsilon_t = \pm 1\}$$

where $k \in K$. But then, as $\underline{\underline{A}}^2(G)$ is abelian,

$$b^{p^j} = \prod_t a_{i_t}^{\varepsilon_t p^j g_t}, \text{ since } k^{p^j} = e ,$$

and so K is finitely generated as a normal subgroup of G by $a_1^{p^j}, \ldots, a_n^{p^j}$. It now follows, as before, that G has max-n. Thus the inductive step is complete.

We have shown, then, that we need prove 6.2.1 only in the two cases a) and c). As the proofs in these two cases are very similar, and rather long, we shall give the proof of case c) only and then indicate the alterations which would be required to give a proof of case a).

Suppose, then, that $\underline{\underline{V}} \leq \underline{\underline{T}}_p\underline{\underline{A}}$ and that $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$, where p is prime. Denote $F_5(\underline{\underline{V}})$ by F and let $y_1, \ldots, y_5$ be a free generating set of F. Denote

$$y_1 \text{ by } y$$

$$[y_2, y_3] \text{ by } z_1$$

$$[y_4, y_5] \text{ by } z_2$$

$$\left[z_1, z_1^{y^i}\right] \text{ by } u_i \quad (i \in \underline{Z})$$

$$\left[z_2, z_2^{y^i}\right] \text{ by } v_i \quad (i \in \underline{Z})$$

$$\left[z_1, z_2^{y^i}\right] \text{ by } w_i \quad (i \in \underline{Z}) .$$

Also, let B be defined as $gp\{z_1, z_2, y\}$. Then the major part of the remainder of this section will be devoted to proving that B has max-n .

We note that $\underline{\underline{A}}^2(F)$ is central in F' and of exponent p (whether p be odd or even). Thus $\underline{\underline{A}}^2(F)$ has a natural structure as a $\underline{GF}(p)F$-module $\{$or $\underline{GF}(p)(F/F')$-module, or $\underline{GF}(p)\{gp(y)\}$-module$\}$. We shall have frequent occasion to use this structure or a variant of it.

6.2.A. We shall show that, if p is an odd prime (the case $p = 2$ will be dealt with in 6.2.B), a relation of the following type must hold in F :

$$u \cdot v \cdot w = e ,$$

where

i) $u = \prod_i u_i^{g_i(y)}$,

ii) $v = \prod_i v_i^{h_i(y)}$,

iii) $w = \prod_i w_i^{f_i(y)}$,

iv) $g_i(y), h_i(y), f_i(y)$ are all elements of $\underline{GF}(p)\{gp(y)\}$ and only a finite number of them are non-zero,

v) $f_0(y)$ is non-zero.

Suppose, on the contrary, that no such relation holds. Denote $gp\{z_1, z_2\}$ by A_o and $A_o^{y^i}$ by A_i. Define $N \trianglelefteq B$ by $N = nsgp_B\{u_i, v_i, w_j \mid i, j \in \underline{Z}, j \neq 0\}$. Since we have assumed that no relation of the above type holds in F, $w_o^{y^i} \notin N$ for all $i \in \underline{Z}$ and so A_iN/N is non-abelian and its centre is generated by $w_o^{y^i}N$. Also, it is an elementary consequence of the definition of N that the elements of $\{A_iN/N \mid i \in \underline{Z}\}$ centralise each other.

We shall show that $gp(\{A_iN/N \mid i \in \underline{Z}\})$ is the direct product of the A_iN/N. It will evidently be sufficient to show that

$$(A_0N/N) \cap (A_1N/N \ldots A_nN/N)$$

is trivial for each n. Since the A_iN/N centralise each other, this intersection is central in both A_0N/N and $(A_1N/N \ldots A_nN/N)$. Now it is easily seen that the centre of a product of subgroups which centralise each other is the product of their centres and so the above intersection is also

$$Z(A_0N/N) \cap \left(Z(A_1N/N) \ldots Z(A_nN/N)\right) .$$

Thus, since $Z(A_iN/N)$ is generated by $w_0^{y^i} N$, if this intersection were non-trivial there would be elements $i_0, \ldots, i_n$ of $\underline{GF}(p)$ such that

non-

$\hat{6}/\hat{6}$

$$w_0^{i_0} N = w_{\cancel{1}}^{i_1 y} \ldots w_{\cancel{n}}^{i_n y^n} N$$

which would yield a relation $u \cdot v \cdot w = e$ of the type above, which, for the time being, we are excluding.

Thus $gp(\{A_iN/N \mid i \in \underline{Z}\})$ is the direct product of the A_iN/N and so $B/N \cong A_0N/N \text{ wr } C(\infty)$. But A_0N/N, being a non-abelian group of $\underline{\underline{T}}_p$, generates $\underline{\underline{T}}_p$ (in fact $A_0N/N \cong T_p$), and so, by 2.3.7, B/N generates $\underline{\underline{T}}_p\underline{\underline{A}}$. But then $\underline{\underline{C}}_p \leq \underline{\underline{T}}_p\underline{\underline{A}} \leq \underline{\underline{V}}$, which contradicts the hypotheses of the proposition. Thus a relation, $u \cdot v \cdot w = e$, of the type described above does, in fact, hold in F.

6.2.B. Suppose now that $p = 2$. We shall show that a relation of the following type must hold in F:

$$u . v . w = e$$

where

i) $u = \prod_i u_i^{g_i(y)} \left(z_1^2\right)^{g(y)}$,

ii) $v = \prod_i v_i^{h_i(y)} \left(z_2^2\right)^{h(y)}$,

iii) $w = \prod_i w_i^{f_i(y)}$,

iv) $g_i(y)$, $g(y)$, $h_i(y)$, $h(y)$, $f_i(y)$ are all elements of $\underline{GF}(2)\left(gp(y)\right)$ and only a finite number of them are non-zero,

v) $f_0(y)$ is non-zero.

Define A_i $(i \in \underline{Z})$ as in 6.2.A and define $M \trianglelefteq B$ by $M = nsgp_B\left(\left\{u_i, v_i, w_j, z_1^2, z_2^2 \mid i, j \in \underline{Z}, j \neq 0\right\}\right)$. If we assume that no relation of the above type holds in F, then $w_0^{y^i} \notin M$ for all $i \in \underline{Z}$ and so A_iM/M is non-abelian. Also A_iM/M is generated by two elements of order 2 and is a group of $\underline{\underline{T}}_2$. Thus A_iM/M is isomorphic to the dihedral group of order 8. ~~and, as in Chapter 5, our definition of T_2 allows the interpretation that $A_iM/M \cong T_2$~~. With these comments, we may now continue, as in 6.2.A, to show that a relation of the above type does, in fact, hold in F .

6.2.C. Let α denote the endomorphism of F defined by

$$\alpha : y_2 \mapsto e$$

$$\alpha : y_i \mapsto y_i \quad i = 1, 3, 4, 5 .$$

Then, whether p is odd or even, $u\alpha = w\alpha = e$ and $v\alpha = v$. Thus $e = e\alpha = (uvw)\alpha = v$. Similarly, $u = e$, and so $w = e$. Thus, whether p is odd or even (and it will no longer be necessary to distinguish these cases), there is a relation in F :

$$\prod_i w_i^{f_i(y)} = e$$

where $f_i(y) \in \underline{GF}(p)\{gp(y)\}$, $f_0(y) \neq 0$.

6.2.D. We recall that $B = gp\{z_1, z_2, y\}$. Denote

w_i $nsgp_B\{z_1, z_2\}$ by B_1 and $nsgp_B(\{u_i, v_i, w_i \mid i \in \underline{Z}\})$ by B_2 .
Since B_2 is normalised by y , it has a natural $\underline{GF}(p)\{gp(y)\}$-module structure. Let T be the torsion submodule of B_2 with respect to this module structure; that is, let

$$T = \left\{a \in B_2 \mid a^{f(y)} = e \text{ for some non-zero element } f(y) \text{ of } \underline{GF}(p)\{gp(y)\}\right\} .$$

Since $\underline{GF}(p)\{gp(y)\}$ is an integral domain, the subset T so defined is in fact a submodule, $T \trianglelefteq B$ and B_2/T is torsion-free as a $\underline{GF}(p)\big(gp(y)\big)$-module. Denote $\mathrm{nsgp}_{B_2/T}\big(\{w_iT \mid i \in \underline{Z}\}\big)$ by L. Then L is also a torsion-free $\underline{GF}(p)\big(gp(y)\big)$-module. In order to avoid becoming bogged down in a morass of notation we shall write the element w_iT of L as w_i; we shall make it clear when we revert to considering w_i as an element of F.

6.2.E. Consider the endomorphism σ of F defined by

$$\sigma : y_4 \mapsto y_4^y$$

$$\sigma : y_5 \mapsto y_5^y$$

$$\sigma : y_i \mapsto y_i \quad i = 1, 2, 3 .$$

Now $u_i\sigma = u_i$, $v_i\sigma = v_i^y$, $w_i\sigma = w_{i+1}$ and $y\sigma = y$. Thus $B_2\sigma \leq B_2$ and so, since $y\sigma = y$, $T\sigma \leq T$. Also σ leaves $\{w_i \mid i \in \underline{Z}\}$ invariant and so σ induces an endomorphism of L which commutes with the action of y.

Consider, now, the polynomial ring $\underline{GF}(p)[\sigma, y]$ in the two commuting indeterminates σ and y. We may define a $\underline{GF}(p)[\sigma, y]$-structure on L by $a^y = a^y$ and $a^\sigma = a^\sigma$ for all $a \in L$. Denote the annihilator of w_0 in $\underline{GF}(p)[\sigma, y]$ by M. Then M is an ideal of $\underline{GF}(p)[\sigma, y]$ which, since the non-trivial relation $\prod_i w_i^{f_i(y)} = e$ holds in F, contains a non-zero element $\sum f_i(y)\sigma^i$. We also note that, since L is torsion-free as a

$\underline{GF}(p)\{gp(y)\}$-module, it follows that, if $h(\sigma, y) \in \underline{GF}(p)[\sigma, y]$ and
f $0 \neq \cancel{h}(y) \in \underline{GF}(p)[y]$ and $f(y)h(\sigma, y) \in M$, then $h(\sigma, y) \in M$.

Denote by $g(\sigma, y) = \sum_{i=0}^{n} g_i(y)\sigma^i$ an element of M with least degree in σ . Evidently, by the remark we have just made, we may assume that if $g(\sigma, y)$ is divisible by $g(y) \in \underline{GF}(p)[y]$, then $g(y)$ is a unit in $\underline{GF}(p)[y]$. Now it follows from a general property of polynomial rings that, if $h(\sigma, y)$ is an arbitrary element of $\underline{GF}(p)[\sigma, y]$, then there exist elements $h(y), k(\sigma, y)$ and $f(\sigma, y)$, where the degree of $f(\sigma, y)$ in σ is less than that of $g(\sigma, y)$, such that

$$h(y)h(\sigma, y) = k(\sigma, y)g(\sigma, y) + f(\sigma, y) .$$

If $h(\sigma, y) \in M$, then $f(\sigma, y) = 0$, by the minimality of $g(\sigma, y)$. Hence, in particular, $h(y) \mid k(\sigma, y)g(\sigma, y)$ and so $h(y) \mid ak(\sigma, y)$ for some $a \in \underline{GF}(p)$, by what we have said above. Thus, whenever $h(\sigma, y) \in M$ there exists an element $k'(\sigma, y) \in \underline{GF}(p)[\sigma, y]$ such that

$$h(\sigma, y) = k'(\sigma, y)g(\sigma, y) .$$

6.2.F. Consider the endomorphism τ_ℓ of F defined by

$$\tau_\ell : y_1 \mapsto y_1^\ell$$

$$\tau_\ell : y_i \mapsto y_i \quad i = 2, 3, 4, 5$$

where $\ell \in \underline{N}$. Then $u_i\tau_\ell = u_{i\ell}$, $v_i\tau_\ell = v_{i\ell}$ and $w_i\tau_\ell = w_{i\ell}$. Thus $B_2\tau_\ell \leq B_2$ and so $T\tau_\ell \leq T$. Thus τ_ℓ induces an endomorphism of L and we have, if $\prod_i w_i^{h_i(y)} \in L$,

$$\left(\prod_i w_i^{h_i(y)}\right)\tau_\ell = \prod_i w_{i\ell}^{h_i(y^\ell)} .$$

Hence, if $h(\sigma, y) \in M$, then $h(\sigma^\ell, y^\ell) \in M$ for all $\ell \in \underline{N}$.

6.2.G. We shall show - and this rather technical detail is the key point of the proof of the proposition - that, for some $a \in \underline{GF}(p)$ and $m \in \underline{N}$, $g_n(y) = ay^m$. The first step will be to show that $(y-1) \nmid g_n(y)$. For, if we suppose that $(y-1) \mid g_n(y)$, we shall show it follows that $(y-1) \mid g_i(y)$ whenever $0 \le i \le n$ and so that $(y-1) \mid g(\sigma, y)$, contrary to our supposition in 6.2.E.

By what we have shown in 6.2.E and 6.2.F, there exist elements $k_\ell(\sigma, y)$ $(\ell \in \underline{N})$ such that

$$g(\sigma^\ell, y^\ell) = k_\ell(\sigma, y)g(\sigma, y) .$$

(We note, for future reference, that by equating the coefficients of the highest powers of σ in these expressions, we obtain that $g_n(y) \mid g_n(y^\ell)$ for all $\ell \in \underline{N}$.) Choose ℓ to be greater than n and not divisible by p; $np + 1$ say. Denote $(\ell-1)n$ by t and write $k_\ell(\sigma, y) = \sum_{i=0}^{t} k_i(y)\sigma^i$.

Equating coefficients of σ^{t+n} in the equation above, we obtain

$$g_n(y^\ell) = k_t(y)g_n(y) .$$

We claim that $(y-1) \nmid k_t(y)$. For, write $g_n(y) = (y-1)^s h(y)$, where $(y-1) \nmid h(y)$. Then

$$g_n(y^\ell) = (y^\ell-1)^s h(y^\ell)$$

$$= k_t(y)(y-1)^s h(y) .$$

Thus $(1 + \ldots + y^{\ell-1})^s h(y^\ell) = k_t(y)h(y)$. Since ℓ is not a multiple of p , $(y-1) \nmid (1 + \ldots + y^{\ell-1})^s$. Hence $(y-1) \nmid k_t(y)$ as, otherwise, $(y-1) \mid h(y^\ell)$ and so $(y-1) \mid h(y)$, contrary to supposition.

We shall now show, by a downwards induction, that $(y-1) \mid g_i(y)$ whenever $0 \le i \le n$. If $i = n$, this is simply the hypothesis and so we may suppose that $0 \leqslant i < n$ and that $(y-1) \mid g_{n-j}(y)$ whenever $0 \le j \le i$. Then, by equating coefficients of $\underset{\sim}{\sigma}^{t+n-i-1}$ in the equation

$$g(y^\ell, \sigma^\ell) = k_\ell(\sigma, y)g(\sigma, y) ,$$

we obtain

$$0 = k_t(y)g_{n-i-1}(y) + k_{t-1}(y)g_{n-i}(y) + \ldots + k_{t-i-1}(y)g_n(y) .$$

Hence $(y-1) \mid k_t(y)g_{n-i-1}(y)$ and so $(y-1) \mid g_{n-i-1}(y)$, completing the inductive step. Hence $(y-1) \mid g_i(y)$ $(0 \le i \le n)$ if $(y-1) \mid g_n(y)$. Thus $(y-1) \nmid g_n(y)$.

Now, as we noted above, $g_n(y) \mid g_n(y^{\ell})$ for all $\ell \in \underline{N}$. Let Ω be the set of roots of $g_n(y)$ in its splitting field over $\underline{GF}(p)$. Then, if $a \in \Omega$, $a^{\ell} \in \Omega$ for all $\ell \in \underline{N}$. However, Ω is a finite set and so, for some integers $r > s$, $a^r = a^s$. Then, unless $a = 0$, $a^{r-s} = 1$ is also a root of $g_n(y)$, which, of course, is not consistent with the already proven fact that $(y-1) \nmid g_n(y)$. Thus all the roots of $g_n(y)$ are zero and so, for some $a \in \underline{GF}(p)$ and $m \in \underline{N}$, $g(y) = ay^m$, as we set out to prove. Returning to the notation of 6.2.D, this implies that L has a relation $w_n^{-ay^m} = \prod_{i=0}^{n-1} w_i^{g_i(y)}$ or, by a slight rearrangement,

$$w_n = \prod_{i=0}^{n-1} w_i^{f_i(y)}$$

where $f_i(y) = -a^{-1}y^{-m}g_i(y)$.

6.2.H. We shall now use this relation to show that B has max-n. We note that we shall now consider the w_i as elements of F, rather than of B/T. The relation above implies that

$$w_n = \prod_i w_i^{f_i(y)} t_1 \quad (t_1 \in T).$$

Also, as the endomorphism of F which maps y to y^{-1} and leaves the other free generators fixed leaves T invariant,

$$w_{-n} = \prod_i w_{-i}^{f_i(y^{-1})} t_2 \quad (t_2 \in T) .$$

Now t_1 and t_2 are elements of T, the $\underline{GF}(p)\{gp(y)\}$ torsion submodule of B_2. Hence there is a non-trivial element $h(y)$ of $\underline{GF}(p)\{gp(y)\}$ such that $t_1^{h(y)} = t_2^{h(y)} = e$. Let T_1 denote the set $\{t \in T \mid t^{h(y)} = e\}$ of elements of B_2 annihilated by $h(y)$. Then $t_1, t_2 \in T_1$ and $T_1 \trianglelefteq B$.

Thus a relation

$$w_n T_1 = \prod_{i=0}^{n-1} w_i^{f_i(y)} T_1$$

holds in F. Recall the definition of the endomorphism σ of F. As this leaves B_2 and y invariant, it will also leave T_1 invariant. Hence applying σ^j $(j \geq 0)$ to this relation we obtain

$$w_{n+j} T_1 = \prod_{i=0}^{n-1} w_{i+j}^{f_i(y)} T_1 \quad (j \geq 0) .$$

Similarly we obtain relations

$$w_{-n-j} T_1 = \prod_{i=0}^{n-1} w_{-i-j}^{f_i(y^{-1})} T_1 \quad (j \geq 0) .$$

Thus, for all integers k,

$$w_k T_1 \in \operatorname{nsgp}_{B/T_1}\left(\{w_i T_1 \;\middle|\; |i| < n\}\right)$$

and so, for each integer k, there are elements $f_{i,k}(y)$ such that

$$w_k T_1 = \prod_{i=-n+1}^{n-1} w_i^{f_{i,k}(y)} T_1 .$$

Consider the endomorphism ρ of F defined by

$$\rho : y_4 \mapsto y_2$$

$$\rho : y_5 \mapsto y_3$$

$$\rho : y_i \mapsto y_i \quad i = 1, 2, 3 .$$

Then $B_2\rho \le B_2$ and $y\rho = y$; so $T_1\rho \le T_1$, and

$$u_k T_1 = (w_k T_1)\rho = \left(\prod_i w_i^{f_{i,k}(y)} T_1\right)\rho$$

$$= \prod_{i=-n+1}^{+n-1} u_i^{f_{i,k}(y)} T_1$$

for each integer k. Thus, in particular,

$$u_k T_1 \in \mathrm{nsgp}_{B/T_1}\left(\{u_i T_1 \;\middle|\; |i| < n\}\right) .$$

Similarly,

$$v_k T_1 \in \mathrm{nsgp}_{B/T_1}\left(\{v_i T_1 \;\middle|\; |i| < n\}\right) .$$

Thus,

$$B_2/T_1 = \text{nsgp}_{B/T_1}\left(\{u_i, v_i, w_i \mid i \in \underline{Z}\}\right)$$

$$= \text{nsgp}_{B/T_1}\left(\{u_i, v_i, w_i \;\middle|\; |i| < n\}\right)$$

and so B_2/T_1 is finitely generated as a normal subgroup of B/T_1.

Hence B_2/T_1 is a finitely generated $\underline{GF}(p)\big(gp(y)\big)$-module. But, by Theorem 1 of [8], the latter has maximal condition on its ideals. Hence B_2/T_1 has maximal condition on its $\underline{GF}(p)\big(gp(y)\big)$-submodules. That is B_2/T_1 has max-B/T_1. But B/B_2 is a finitely generated metabelian group and so, by 2.2.6, has max-n. Thus, by 2.2.5, B/T_1 has max-n.

As B_2/T_1 has maximal condition on its $\underline{GF}(p)\big(gp(y)\big)$-submodules, the torsion submodule T/T_1 is 'bounded'; that is, there is an element, $f(y)$, of $\underline{GF}(p)\big(gp(y)\big)$ such that, if $t \in T$, $t^{f(y)} \in T_1$. But then, recalling the definition of $h(y)$, $t^{f(y)h(y)} = e$. Let $g(y) = h(y)f(y)$ and denote by S the set $\left\{b^{g(y)} \mid b \in B_2\right\}$. We claim that $S \cap T = E$. For, let $b \in S \cap T$. Then, since $b \in S$, $b = a^{g(y)}$ for some element a of B_2. Also, since $b \in T$, $a^{g(y)} \in T$ and so $a \in T$. Thus $b = a^{g(y)} = e$. (This result is evidently a module analogue of 2.2.4.) Now $S \trianglelefteq B$ and, since $T \geq T_1$, B/T has max-n. Thus to show that B has max-n it will be sufficient, by 2.2.5, to show that B/S has max-n.

Consider $gp\big(B_2/S, yS\big) \in \underline{\underline{A_p A}}$. Recalling the definition of S, it is evident that this group has a law of the form $[x_1, x_2]^{k(x_3)} d$ $(d \in \underline{\underline{A}}_p\underline{\underline{A}}(X))$ where $k(x_3)$ is a non-zero element of $\underline{GF}(p)\big(gp(x_3)\big)$. Thus, by 2.3.9, $\underline{\underline{A}}_p\underline{\underline{A}} \neq \operatorname{var}\big(gp(B_2/S, yS)\big)$ and so, by 2.3.5, y^nS centralises B_2/S for some natural number n . Denote $gp\big(B_1, y^n\big)$ by D . Then $|B : D| = n$ and so, since B is finitely generated, D is finitely generated. Also, $D/B_2 \in \underline{\underline{A}}^2$ and B_2/S is central in D/S . Thus $D/S \in [\underline{\underline{E}}, \underline{\underline{A}}^2]$. By the hypothesis of the theorem, $\underline{\underline{C}}_p \nsubseteq \operatorname{var}(D/S)$ and so, by 5.1.1 and 2.2.6, D/S has max-n . Thus, since $|B : D|$ is finite, B/S also has max-n . Hence B has max-n , as we set out to prove.

REMARK. We have here used for the first time the assumption that $\underline{\underline{C}}_p \nsubseteq \underline{\underline{V}}$. Previously the proof depended only on the assumption that $\underline{\underline{T}}_p\underline{\underline{A}} \nsubseteq V$.

6.2.I. As B has max-n , $nsgp_B(\{w_i \mid i \in \underline{Z}\})$ is finitely generated as a normal subgroup of B . We may suppose, then, that it is generated as a normal subgroup of B by $\{w_i \mid |i| < m\}$ for some natural number m . Thus there are elements $g_{i,k}(y)$ of $\underline{GF}(p)\big(gp(y)\big)$ such that

$$w_k = \prod_{i=-m}^{m} w_i^{g_{i,k}(y)} \qquad k \in \underline{Z} .$$

From these relations in the free generators of a free group of $\underline{\underline{V}}$

(recall the definition of w_i and y), we deduce the following laws of $\underline{\underline{V}}$

$$\left[[x_2, x_3], [x_4, x_5]^{x_1^k}\right]$$

$$= \prod_{i=-m}^{m} \left[[x_2, x_3], [x_4, x_5]^{x_1^i}\right]^{g_{i,k}(x_1)} \qquad k \in \underline{Z} .$$

Suppose that G is a finitely generated group in $\underline{\underline{V}}$ and that $a, b \in G'$, $g \in G$. Now the law above is a law in G; substitute a for x_2, b for x_4 and g for x_1, x_3 and x_5. Then, noting that

$$\left[[a, g], [b, g]^{g^k}\right] = \left[a, b^{g^k}\right]^{g+1} \left[a, b^{g^{k+1}}\right]^{-1} \left[a, b^{g^{k-1}}\right]^{-g} ,$$

it is elementary, but tedious (and so we omit the proof) to show that these relations imply that

$$\left[a, b^{g^k}\right] \in \operatorname{nsgp}_G\left(\left\{\left[a, b^{g^i}\right] \mid |i| \leq m\right\}\right)$$

for each integer k.

Now $G/\underline{\underline{A}}^2(G)$ is a finitely generated metabelian group and so, by 2.2.6, has max-n. Thus $G'/\underline{\underline{A}}^2(G)$ is finitely generated as a normal subgroup of G/G'' - by $a_1G'', \ldots, a_kG''$ say. Thus $\underline{\underline{A}}^2(G)$ is generated as a normal subgroup of G by

$$\left\{\left[a_i, a_j^g\right] \mid 1 \leq i \leq j \leq k, g \in G\right\} .$$

To prove that G has max-n it will evidently be sufficient to show that G'' is finitely generated as a normal subgroup of G and so, to show that, for each pair i, j $(1 \le i \le j \le k)$,
$\mathrm{nsgp}_G\left(\left\{\left[a_i, a_j^g\right] \mid g \in G\right\}\right)$ is finitely generated as a normal subgroup of G. Thus the proof of case c) of Proposition 6.2.1 will follow from:

6.2.2 LEMMA. *Let* G *be a finitely generated group of* $\underline{\underline{N}}_c\underline{\underline{N}}_d$ $(c \ge 2)$ *and denote* $\underline{\underline{N}}_d(G)$ *by* N. *Suppose that, for some natural number* m, *the following condition is satisfied:*

whenever $a \in N$, $b \in \underline{\underline{N}}_{c-2}(N)$ *(or* N, *if* $c = 2$*) and* $g \in G$, $\left[a, b^{g^k}\right] \in \mathrm{nsgp}_G\left(\left\{\left[a, b^{g^i}\right] \mid |i| \le m\right\}\right)$ *for each integer* k.

Then, if a, b *are as above, there is a finite subset* Ω $(= \Omega(a, b))$ *of* G *such that*

$$\mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in G\right\}\right) \le \mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in \Omega\right\}\right).$$

(NOTE. This lemma is, of course, much more general than we require here. We shall, however, find occasion to use the full generality in Section 6.3).

Proof. As G/N is a finitely generated nilpotent group, it is polycyclic. Thus there is a series

$$N = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_j \trianglelefteq \ldots \trianglelefteq H_r = G$$

of subgroups of G such that H_{j+1}/H_j is cyclic. Hence we may choose elements $g_1, \ldots, g_r$ of G such that $H_j = gp(g_1, \ldots, g_j, N)$ $(1 \le j \le r)$. We shall prove, by induction on j, that for each j, and each pair of elements a, b $(a \in N,\ b \in \underline{\underline{N}}_{c-2}(N))$ there is a finite subset Ω_j $(= \Omega_j(a, b))$ of G such that

$$\mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in H_j\right\}\right) = \mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in \Omega_j\right\}\right).$$

If $i = 0$, $\mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in H_0\right\}\right) = \mathrm{nsgp}_G([a, b])$ as, for every element d of N, $[a, b^d] = [a, b]$. Thus we may take $\Omega_0(a, b) = \{e\}$ for every pair a, b. Suppose, then, that for some $j \ge 1$, we have demonstrated the existence of the sets Ω_{j-1}. Then

$$\mathrm{nsgp}_G\left(\left\{[a, b^h] \mid h \in H_j\right\}\right)$$

$$= \mathrm{nsgp}_G\left(\left\{\left[a, b^{hg^i}\right] \mid h \in H_{j-1},\ i \in \underline{Z}\right\}\right),$$

where $g = g_j$,

$$= \mathrm{nsgp}_G\left(\left\{\left[a, b^{hg^i}\right] \mid h \in H_{j-1},\ |i| \le m\right\}\right),$$

by the initial hypothesis of the lemma,

$$= \mathrm{nsgp}_G\left(\left\{\left[a, b^{g^i h}\right] \mid h \in H_{j-1},\ |i| \le m\right\}\right),$$

as $H_{j-1} \trianglelefteq H_j$,

$$= \operatorname{nsgp}_G\left(\left\{\left[a,\ b^{g^i h}\right] \mid h \in \Omega'_{j-1},\ |i| \le m\right\}\right),$$

where $\Omega'_{j-1} = \bigcup_i \Omega_{j-1}\left(a,\ b^{g^i}\right)$, $|i| \le m$,

$$= \operatorname{nsgp}_G\left(\left\{\left[a,\ b^h\right] \mid h \in \Omega_j(a,\ b)\right\}\right),$$

where $\Omega_j(a,\ b) = \left\{g^i h \mid h \in \Omega'_{j-1},\ |i| \le m\right\}$. Thus, as $\Omega_j(a,\ b)$ is finite, the inductive step is complete. Thus, by taking $\Omega(a,\ b) = \Omega_r(a,\ b)$, the proof of the lemma follows.

Thus the proof in case c) is complete. For the proof in case a), apart from substituting $\underline{Z}$ for $\underline{GF}(p)$ throughout, there are only three points at which the proof we have given requires substantial alteration. Firstly, consider 6.2.A. Then, in case a), $A_i N/N$ is a 2-generator, non-abelian, torsion-free group of $\underline{\underline{N}}_2$ and so $A_i N/N \cong F_2\left(\underline{\underline{N}}_2\right)$ and $Z\left(A_i N/N\right) = \left(A_i N/N\right)'$ is generated by $w_0^{y^i} N$. With these remarks the proof in 6.2.A now carries over to case a).

Secondly, consider 6.2.G. The 'slight rearrangement' of the elements $g_i(y)$ in the last sentence of that section depends on the fact that a is invertible. In case a), when $a \in \underline{Z}$, this, of course, requires proof. Note that a may be taken positive, and then it is the greatest integral divisor of $g_n(y)$. Assume $a > 1$ and consider the equations we used in 6.2.G to show that $(y-1) \nmid g_n(y)$. Then a is also the greatest integral divisor of

$g_n(y^{\ell})$ and so $k_t(y)$ is primitive. We may now continue, as with the proof that $(y-1) \nmid g_n(y)$, to show that $a \mid g(\sigma, y)$ - an evident contradiction. Hence $a = 1$ and the proof in 6.2.G carries over to the case a).

Finally, the proof in 6.2.H, that B/S has max-n, requires alteration. We recall that $g(y)$ satisfies $t^{g(y)} = e$, whenever $t \in T$. As, in case a), F is torsion-free, we may suppose that $g(y)$ is a primitive element of $\underline{Z}(gp(y))$ - that is, the coefficients of powers of y have no non-unit common factor in $\underline{Z}$. Thus $[x_1, x_2]^{g(x_3)}$ is a law of $gp(B_2/S, yS)$ and so, by i) of 2.3.9, $\underline{\underline{A}}_p\underline{\underline{A}} \nleq \mathrm{var}\left(gp(B_2/S, yS)\right)$ for all primes p. Hence, by 2.3.3 and 2.3.4, $gp(B_2/S, yS) \in \underline{\underline{N}}_c\underline{\underline{A}}_n$ for some natural numbers c and n. Thus, if $b \in B_2$, $b^{(y^n-1)^c} \in S$. Denote $gp\left(B_1, y^n\right)$ by D. Then $|B : D| = n$ and so D is finitely generated. Also $\underline{\underline{A}}^2(D) \leq B_2$, and since $B_2^{(y^n-1)^c} \leq S$, $D/S \in [\underline{\underline{A}}^2, c\underline{\underline{E}}]$. Thus repeated application of 5.1.1, shows that D/S, and so B/S, is abelian by nilpotent by finite exponent and so, by 2.2.6, B/S has max-n.

Apart from these alterations, the proof of case a) is substantially the same as the one we have given. The proof of Proposition 6.2.1 is therefore complete.

6.3 Metanilpotent varieties

In this section, we shall extend the result of 6.2.1 to all metanilpotent varieties. We shall also show that a similar result holds when 'max-n' is replaced by 'residual finiteness'.

6.3.1 THEOREM. *Let* $\underline{\underline{V}}$ *be a metanilpotent variety. Then the finitely generated groups of* $\underline{\underline{V}}$ *have* max-n *if and only if* $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ *for all primes* p .

Proof. If $\underline{\underline{C}}_p \leq \underline{\underline{V}}$ for some prime p , then $\underline{\underline{V}}$ contains an uncountable number of 2-generator groups, by Theorem 8 of [8] and so, in particular, $F_2(\underline{\underline{V}})$ does not have max-n .

Suppose, then, that $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ for all primes p and that $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{N}}_d$ for some natural numbers c and d . We shall argue by induction on c . If $c = 1$, the result follows from 2.2.6 and so we may suppose that $c > 1$ and that, for all d , if H is a finitely generated group of $\underline{\underline{V}} \wedge \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d$, then H has max-n .

We shall require some notation:

let F denote $F_\infty(\underline{\underline{V}})$;

let $y_1, \ldots, y_k, \ldots$ be a free generating set of F ;

let w denote

$\hat{\ell}$ $$\big[[y_1, \ldots, y_{d+1}], [y_{d+2}, \ldots, y_{2d+2}], \ldots, [y_m, \ldots, y_n]\big]$$

ℓ where $m = (c-2)(d+1) + 1$ and $\cancel{n} = (c-1)(d+1)$;

let z denote $[y_{(c-1)(d+1)+1}, \ldots, y_{c(d+1)}]$;

let y denote $y_{c(d+1)+1}$.

Then $z \in \underline{\underline{N}}_d(F)$ and $w \in \underline{\underline{N}}_{c-2}\big(\underline{\underline{N}}_d(F)\big)$.

Denote $gp(w, z, y)$ by H and $gp\left(\left\{w^{y^i}, z^{y^i} \mid i \in \underline{Z}\right\}\right)$ by H_1 . Then $H_1 \trianglelefteq H$. Suppose, for the time being, that $c > 2$. Then, by 1.7 of [9], $\underline{\underline{N}}_{c-1}(H_1)$ is generated by all left-normed commutators of weight c in the generators of H_1 . A commutator of this kind vanishes if at least one of its entries is a conjugate of w : for $w \in \underline{\underline{N}}_{c-2}\big(\underline{\underline{N}}_d(F)\big)$, $z \in \underline{\underline{N}}_d(F)$, $c \geq 3$ and $\underline{\underline{N}}_c\big(\underline{\underline{N}}_d(F)\big) = E$. Hence $\underline{\underline{N}}_{c-1}(H_1)$ is generated by commutators involving only conjugates of z . Now $H/\underline{\underline{N}}_{c-1}(H_1)$ is a finitely generated group of $\underline{\underline{V}} \wedge \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d$ and so, by our inductive hypothesis, has max-n . Hence

$$nsgp_{H/\underline{\underline{N}}_{c-1}(H_1)}\left(\left\{\left[z, w^{y^k}\right]\underline{\underline{N}}_{c-1}(H_1) \mid k \in \underline{Z}\right\}\right)$$

is finitely generated as a normal subgroup of $H/\underline{\underline{N}}_{c-1}(H_1)$ and so there is a natural number n , and, for each integer k , an element $f_{i,k}(y)$ of $\underline{Z}\big(gp(y)\big)$ and an element u_k of $\underline{\underline{N}}_{c-1}(H_1)$ such that

$$\left[z, w^{y^k}\right] = u_k \prod_{i=-n}^{+n} \left[z, w^{y^i}\right]^{f_{i,k}(y)} .$$

Consider the endomorphism τ of F defined by

$$\tau : y_i \mapsto e \quad \text{if} \quad i = 1$$

$$\tau : y_i \mapsto y_i \quad \text{if} \quad i > 1 \, .$$

Then $w\tau = e$, $z\tau = z$ and $y\tau = y$. Also $u_k\tau = u_k$ (recall that u_k is a product of commutators involving only powers of z under conjugates of powers of y). Thus, on applying τ to the relations above, we find that $u_k = e$. If $c = 2$, then, by 6.2.1, H has max-n and so relations of the type above (with $u_k = e$) again hold.

Now w, z and y involve non-overlapping subsets of the free generating set of F and so we may, independently, replace w, z or y by endomorphic images in F (the technique is similar, for example, to that used in the proof of 2.3.1). But we have chosen w, z and y so that their endomorphic images generate $\underline{\underline{N}}_{c-2}\left(\underline{\underline{N}}_d(F)\right)$, $\underline{\underline{N}}_d(F)$ and F respectively - in fact, in the last case every element of F is an endomorphic image of y . To avoid the Slough of Notation, we suppress the otherwise easy proof of the fact that this implies that, if $a \in \underline{\underline{N}}_d(F)$ and $b \in \underline{\underline{N}}_{c-2}\left(\underline{\underline{N}}_d(F)\right)$ and $g \in F$,

$$\operatorname{nsgp}_F\left(\left\{\left[a, b^{g^k}\right] \mid k \in \underline{Z}\right\}\right) = \operatorname{nsgp}_F\left(\left\{\left[a, b^{g^i}\right] \;\middle|\; |i| \le n\right\}\right) .$$

Also, as every finitely generated group G of $\underline{\underline{V}}$ is a homomorphic image of F , such relations will hold in G .

Denote $\underline{\underline{N}}_{c-1}(\underline{\underline{N}}_d(G))$ by L. By our induction hypothesis, G/L, being a finitely generated group of $\underline{\underline{V}} \wedge \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d$, has max-n. Thus $\underline{\underline{N}}_d(G)/L$ and $\underline{\underline{N}}_{c-2}(\underline{\underline{N}}_d(G))/L$ are finitely generated as normal subgroups of G/L - by $a_1L, \ldots, a_rL$ and $b_1L, \ldots, b_sL$, respectively, say. Thus, as $L = [\underline{\underline{N}}_d(G), \underline{\underline{N}}_{c-2}(\underline{\underline{N}}_d(G))]$, L is generated as a normal subgroup of G by

$$\left\{\left[a_i, b_j^g\right] \mid 1 \leq i \leq r, 1 \leq j \leq s, g \in G\right\}$$

(recall that L is central in $\underline{\underline{N}}_d(G)$). Hence, by 6.2.2, L is finitely generated as a normal subgroup of G. But L is central in $\underline{\underline{N}}_d(G)$ and so $G/C_G(L)$ is nilpotent. Hence, by Theorem 1 of [8], $\underline{Z}(G/C_G(L))$ has maximal condition on its right ideals and so L has max-G. Thus, by 2.2.5, G has max-n and the proof of Theorem 6.3.1 is complete.

6.3.2 THEOREM. *Let $\underline{\underline{V}}$ be a metanilpotent variety. Then the finitely generated groups of $\underline{\underline{V}}$ are residually finite if and only if $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ for all primes* p.

Proof. If $\underline{\underline{C}}_p \leq \underline{\underline{V}}$ for some prime p, then T_p cr $C(\infty) \in \underline{\underline{V}}$ and we have shown, as part of the proof of 5.2.4, that T_p cr $C(\infty)$ is finitely generated but not residually finite.

Suppose, then, that $\underline{\underline{C}}_p \nleq \underline{\underline{V}}$ for all primes p and that $\underline{\underline{V}} \leq \underline{\underline{N}}_c\underline{\underline{N}}_d$ for some natural numbers c and d. It will evidently be sufficient to prove, by induction on c, that if G is a finitely generated monolithic group in $\underline{\underline{V}}$ with monolith M, then G is finite. If $c = 1$, the result follows from Theorem 1 of [11] and so we may suppose that $c > 1$, $G \notin \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d$ and that all finitely generated groups of $\underline{\underline{N}}_{c-1}\underline{\underline{N}}_{d'} \wedge \underline{\underline{V}}$ are, for every $d' \in \underline{N}$, residually finite.

Denote $\underline{\underline{N}}_{c-1}(\underline{\underline{N}}_d(G))$ by Z. Then Z is a non-trivial normal subgroup of G and so $Z \geq M$. Denote $C_G(Z)$ by C and the split extension of Z by G/C, by H. Since G has max-n, Z is finitely generated as a normal subgroup of G and so H is finitely generated. Also, as Z is central in $\underline{\underline{N}}_d(G)$, G/C is nilpotent and so H is abelian by nilpotent. Thus, by Theorem 1 of [11], H is residually finite.

We make the identification necessary to ensure that Z is a normal subgroup of both G and H; the G-normal and H-normal subgroups of Z are then the same. We claim that H is monolithic with monolith M. For suppose that $K \trianglelefteq H$ and that $K \ngeq M$. Since $K \cap Z \trianglelefteq H$, then also $K \cap Z \trianglelefteq G$ and so, since $M \nleq K \cap Z$, $K \cap Z = E$. Thus, as K and Z are normal subgroups of H, $[K, Z] = E$. But Z is a self-centralising normal subgroup of H and so $K = E$. Hence H is a monolithic, residually finite group and so is finite. In particular, G/C is finite and so C is finitely generated, and, of course, Z and M are finite.

Suppose we have shown that all finitely generated groups in $[\underline{\underline{E}}, \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d] \wedge \underline{\underline{V}}$ are residually finite. Let K be a normal subgroup of C maximal with respect to not containing the monolith

M . Then C/K is monolithic. Hence, since C , and so C/K , is a finitely generated group of $[\underline{\underline{E}}, \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d] \wedge \underline{\underline{V}}$, C/K is finite. Thus the normaliser of K in G has finite index in G and so K has only a finite number of conjugates in G . If K_1 is the intersection of all these conjugates, $K_1 \trianglelefteq G$ and G/K_1 is finite. But $M \nleq K_1$; thus $K_1 = E$ and so G is finite in this case.

We may therefore suppose that $G \in [\underline{\underline{E}}, \underline{\underline{N}}_{c-1}\underline{\underline{N}}_d] \wedge \underline{\underline{V}}$; that is, that $C = G$. Since Z is finite, we also have $G \in \underline{\underline{A}}_n\underline{\underline{N}}_{c-1}\underline{\underline{N}}_d$ for some $n \in \underline{N}$. Let k be a natural number greater than $cd + c + 1$ and also greater than the cardinality of some finite generating set of G . Denote $F_k(\text{var } G)$ by F and let $y_1, \ldots, y_k$ be a free generating set of F . Then $\underline{\underline{N}}_{c-1}(\underline{\underline{N}}_d(F)) \in \underline{\underline{A}}_n$. But F is a finitely generated group in $\underline{\underline{V}}$ and so, by 6.3.1, has max-n . Since $\underline{\underline{N}}_{c-1}(\underline{\underline{N}}_d(F))$ is central in F , it is finitely generated; since it is also of finite exponent it is finite.

Denote

$$[y_{i(d+1)+1}, \ldots, y_{(i+1)(d+1)}]$$

by w_i $(0 \le i \le c-1)$ and y_{cd+c+1} by y . Then

$$\left\{\left[w_0, \ldots, w_{c-2}, w_{c-1}^{y^i}\right] \mid i \in \underline{Z}\right\}$$

is a subset of the finite group $\underline{\underline{N}}_{c-1}(\underline{\underline{N}}_d(F))$ and so cannot be infinite. Thus, for some distinct integers r and s ,

$$\left[w_0, \ldots, w_{c-2}, w_{c-1}^{y^r}\right] = \left[w_0, \ldots, w_{c-2}, w_{c-1}^{y^s}\right]$$

and so, for the natural number $t = |r-s|$,

$$\left[w_0, \ldots, w_{c-2}, \left[w_{c-1}, y^t\right]\right] = e .$$

Denote $\underline{\underline{N}}_d(F)$ by N and $\underline{\underline{B}}_t(F)$ by B. Then, as $\underline{\underline{N}}_d$ is defined by the word $[x_1, \ldots, x_{d+1}]$ and $\underline{\underline{B}}_t$ by the word x_1^t, the above relation implies that

$$[N, \ldots, N, [N, B]] = E$$

and so, in particular, that $[N, B]$ has class at most $c - 1$. But $B/N \cap B \cong NB/N$ has class at most d, and $N \cap B/[N, B]$ is central in $B/[N, B]$. Thus $B/\cancel{N \cap B}$ [N,B] is nilpotent of class at most $d + 1$. Hence $B \in \underline{\underline{N}}_{c-1}\underline{\underline{N}}_{d+1}$ and so $F \in \underline{\underline{N}}_{c-1}\underline{\underline{N}}_{d+1}\underline{\underline{B}}_t$. As G is a homomorphic image of F, $G \in \underline{\underline{N}}_{c-1}\underline{\underline{N}}_{d+1}\underline{\underline{B}}_t$ also. Since $G/\underline{\underline{B}}_t(G)$ is finite, $\underline{\underline{B}}_t(G)$ is finitely generated and so, by our inductive hypothesis, is residually finite. Thus, by 26.12 of [19], G is residually finite and so is finite, completing the proof of the theorem.

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p. 2. The last line should read

'variety generated by the relatively free group of rank 2 of the variety of all centre by metabelian groups with derived group of' .

p. 19. The last line of Section 2.2 should read

'T_2 is the dihedral group of order 8' .

p. 21. Line 5 - immediately preceeding 2.3.1 - should read

'$\underline{\underline{C}}_p = \mathrm{var}\left(F_2\left(\underline{\underline{T}}_p\underline{\underline{A}} \wedge [\underline{\underline{E}}, \underline{\underline{A}}^2]\right)\right)$ ' .

p. 71. Insert p. 71a, immediately preceeding p. 71. On page 71, delete the first line and the word 'Proof'.

p. 79. Omit the fourth sentence of the second paragraph, that is, from 'Our definition ...' to '... 5.2.4). '

p. 89. In the third line from the bottom of the page, omit from 'and, as in Chapter 5 ...' to the end of the sentence.

p. 13. 10 Omit the second occurrence of p

7 from $\{(u, v)$ ..... to the end of the sentence should read

'$\{(u, v)\eta \mid u, v \in G/N\}$ where η is the embedding $\eta : N \to N^*$.'

Groves Thesis - Consolidated misprints

Reference	For	Read
2^{8}	dual	
23_{10}	$\underline{\underline{A}}_m\underline{\underline{A}}$	$\underline{\underline{A}}_n\underline{\underline{A}}$
28^{7}	$w\delta\xi'$	$w\delta\xi'(c^{m})$
29^{5}	2.3.8	2.3.7
31^{5}	Varieties	varieties
35^{9}	[1]	[19]
37_{6}	c	d (twice)
38^{6}	[17]	[2]
38_{3}	H	(H)
39^{7}	2.3.11	2.3.10
42_{3}	groups	finite groups
43^{7}	η^{i}	η_n^{i}
43^{10}	δ	δ_n (thrice)
	η	η_n (twice)
$43^{11,12,13}$	$\eta^{i}\delta$	$\eta_n^{i}\delta_n$
43_{1}	$\eta_i\delta$	$\eta_n^{i}\delta_n$
46^{8}	(first) be	by
49^{7}	$\underline{\underline{B}}_m(G)$	$\underline{\underline{B}}_m(G)N$
52_{9}	$\underline{\underline{B}}_k$	$\underline{\underline{B}}_n(k)$
53^{6}	$\underline{\underline{B}}_r(q)$	$\underline{\underline{B}}_r(q)^{(F)}$
57_{4}	endomrophism	endomorphism
59_{5}	for	far
60^{6}	$\underline{\underline{T}}_p$	$\underline{\underline{C}}_p$
62_{7}	A	A_1
63_{2}	2e+1	2e+3
66^{1}	$\underline{\underline{B}}_p$.	$\underline{\underline{B}}_p\underline{\underline{A}}$.
66^{6}	every	every non-zero

66^9	$\ell_i \geq 0$	$\ell_i > 0$
$67^{1,3}$	$\underline{\underline{B}}_4(X_2)$	$\underline{\underline{B}}_4(X_2')$
67^9	$\underline{\underline{B}}_2(X_2)$	$\underline{\underline{B}}_2(X_2')$
67^{10}	$\underline{GF}$	$\underline{GF}(2)$
67_4	$\ell_i \geq 0$	$\ell_i > 0$
69_4	$f_3(y^{-1})$	$-f_3(y^{-1})$
$71a_7$	yB	xB
74^{10}	$p^i(p^i-1)/2$	$p^{i-1}(p^{i-1}-1)/2$
$74_{6,5,3,2}$	N	M
75_3	2	3 (thrice)
$76_{10,8,7}$	2	3
77^1	2	3
79_9	2	3
80^6	2	3
81^1	f	f_3
81^5	$\underline{\underline{B}}_m(F)$	$\underline{\underline{B}}_m(F/Z)$
$81^{7,8,9}$	M	B
88^9	trivial	non-trivial
88^{11}	w_1, w_n	w_0
90_8	m_i	w_i
91^1	B_2/T	B/T
92^2	$h(y)$	$f(y)$
94_9	$0 < i < n$	$0 \leq i < n$
94_7	σ	σ
$97^{2,9}$	$\prod_{i=-n}^{n}$	$\prod_{i=-n+1}^{n-1}$
106_5	align	
105_1, 106^1	n	ℓ
109_2	E	$\underline{\underline{E}}$
111_7	$B/N \cap B$	$B/[N, B]$
113^{13}	Ph.D.	D. Phil.